Self-Fulfilling Risk Panics[1]

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Abstract

Recent crises have seen very large spikes in asset price risk with the VIX at times tripling or quadrupling. We propose an explanation for such risk panics based on self-fulfilling shifts in beliefs about risk. This is the result of a negative link between the current asset price and risk about the future asset price, which gives rise to a circular relationship between the stochastic process of asset price risk and the asset price itself. We find that self-fulfilling shifts in perceived risk can be coordinated both around a pure sunspot as well as a macro fundamental. We also show that when a risk panic is set off by a sunspot trigger event, a macro fundamental can play a key role as a focal point of the market that affects both the magnitude of the panic and subsequent shifts in perceived risk.

1 Introduction

Sharp surges in asset price risk are a prominent feature of financial panics, such as the turmoil in the Fall of 2008 or more recently the European debt crisis. Implied volatility, as measured by the VIX index, more than quadrupled in the wake of the Lehman Brothers failure, and tripled both in May 2010 and August 2011 in connection to the European debt crisis. Explaining such huge and sudden spikes in risk is an important theoretical challenge that the literature has yet to meet. In this paper we propose a theory for large self-fulfilling changes in beliefs about risk.

We frame our analysis in a very simple model where agents have mean-variance preferences and choose to allocate their wealth between a riskfree bond and a risky asset. The key implication is that the equilibrium asset price Q_t depends negatively on asset price risk, defined as uncertainty about the asset price tomorrow, $var_t(Q_{t+1})$. To see how self-fulfilling shifts in risk can arise in this context, assume that agents believe that the risk $var_t(Q_{t+1})$ depends on a variable S_t. This implies that Q_t depends on S_t as well as the asset price depends negatively on risk, and therefore Q_{t+1} depends on S_{t+1}. Now, if we assume that the distribution of S_{t+1} depends on S_t, then the risk $var_t(Q_{t+1})$ will indeed depend on S_t. This circular relationship between asset price risk and the asset price level can therefore generate self-fulfilling shifts in risk coordinated around the variable S_t. What is interesting is that S_t could be a fundamental variable like dividends, but it could also be a variable extrinsic to the model, i.e. a sunspot variable.

We consider three versions of the model, which contrast the various roles of the state variable(s). In the first version there is just one state variable that is a pure sunspot. The risky asset pays a constant dividend. We consider both an autoregressive and a Markov process for the sunpot. We show that there is a fundamental equilibrium where the asset price is constant, and a sunspot equilibrium where the asset price, and asset price risk, fluctuate with the sunspot. Changes in the value of the sunspot can trigger both a large increase in risk and a drop in the asset price that we refer to as a risk panic. In such risk panics the change in perceived risk is entirely self-fulfilling.

In the second version of the model we assume that the state variable affects the dividend of the risky asset, and is therefore a fundamental. We consider the same two processes for the state variable as in the first version. There is again a fundamental equilibrium, where risk is constant and changes in the asset price reflect the impact of the state variable on the dividend. There is also a so-called "sunspot-like" equilibrium in which the state variable plays a dual role. It plays both the role of a fundamental that affects dividends and the role of a sunspot that generates self-fulfilling shifts in risk. The latter role dominates.[1]

Finally, in the third version of the model there are two state variables. The first is a fundamental that affects dividends as in the second version. The second is a two-state sunspot variable. This gives rise to equilibria akin to a switch between the fundamental and sunspot-like equilibria, with the sunspot playing the role of a trigger variable for the switch. A change of the sunspot can be seen as a shift from a non-panic to a panic state, which we call a risk panic. During such a risk panic there is a spike in asset price risk and a drop in the asset price that can be very large in magnitude. We show that the panic is larger when the fundamental is weak, and that the asset price becomes much more sensitive to the fundamental once we shift to the panic state. The role of the fundamental in the panic state is the same as in the sunspot-like equilibria discussed above. In addition to its fundamental role, it becomes a focal point for self-fulfilling changes in beliefs about risk.

The paper is related to the broader literature on multiple equilibria with self-fulfilling shifts in beliefs. In this literature there is a coordination of beliefs among agents, such that a common shift in beliefs leads to actions of all agents that make the change in expectations self-fulfilling. In terms of asset prices, there are many applications of this phenomenon for both stock prices and exchange rates. In particular, there is a large literature with self-fulfilling speculative attacks on currencies.[2] A key distinction here is that the self-fulfilling shift in beliefs is not about the level of the asset price but about the level, and more generally the process, of risk. This is critical as we wish to explain large spikes in risk.

There is also a literature focusing on self-fulfilling shifts in beliefs about risk that are due to an interaction between risk and liquidity. This occurs in limited

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[1] The term "sunspot-like" equilibria was first coined by Manuelli and Peck (1992). They write: "There are two ways that random fundamentals can influence economic outcomes. First, randomness affects resources which intrinsically affects prices and allocation. Second, the randomness can endogenously affect expectations or market psychology, thereby leading to excessive volatility." In the limiting case where fundamental uncertainty goes to zero, sunspot-like equilibria converge to pure sunspot equilibria.

[2] E.g., see Obstfeld (1986), Aghion et al. (2004), or Burnside et al. (2004).

participation models such as Pagano (1989), Allen and Gale (1994) and Jeanne and Rose (2002). When agents believe that risk is high, market participation is low. This implies low market liquidity, which leads to a large price response to asset demand shocks and therefore high risk.[3] This is quite different though from what happens in our model, where there is no concept of market liquidity. In contrast to static limited participation models, the dynamic nature of the model is critical in generating our results. In our setting sunspot equilibria cannot occur in the absence of dynamic relation between the state variable today and its distribution tomorrow.

We derive our results under the assumption that agents have simple mean-variance preferences. The mean-variance portfolio model has a long history in academics and remains extensively used today. It is also widely used in the financial industry and can therefore be considered as a reasonable description of actual behavior.[4] An alternative avenue would be to introduce micro founded risk-based portfolio constraints, such as value-at-risk constraints or margin constraints, so that asset demand (and therefore the asset price itself) would depend explicitly on uncertainty about the future asset price.[5] This would, however, make the model significantly more complicated. The mean-variance portfolio assumption in this paper should then be considered as an approximation of more complex behavior.

The model is too simple to calibrate to actual data of financial panics. However, at a qualitative level it does connect to events in recent years in several ways. First, it can generate spikes in risk and a drop in asset prices that are very large, as we show through numerical illustrations. We are not aware of any other macro model that can generate the huge spikes in risk as seen during the U.S. financial crisis in 2008 or the European debt crisis. Second, this happens without any change in fundamentals. Balance sheets of U.S. financial institutions had started to gradually deteriorate long before the financial panic in the Fall of 2008. The same can be said for Greek debt, which did not suddenly reach its high level in May of 2010,

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[3] This phenomenon is not limited to limited participation models of asset prices. For other applications see Bacchetta and van Wincoop (2006) and Walker and Whiteman (2007).

[4] See Basak and Chabakauri (2009) for further motivation.

[5] A substantial literature introducing such constraints has developed in recent years. Examples are Brunnermeier and Pedersen (2009), Danielsson, Shin and Zigrand (2010) and Gromb and Vayanos (2002). For the same reason of analytic tractability as in this paper, these constraints are often introduced in a reduced-form way rather than based on explicit micro foundations.

when it first ignited a spike in risk. Finally, the last version of our model implies that a risk panic also leads to increased volatility of risk that is coordinated around news about a macro fundamental. During recent market turmoil associated with European debt, any news about Greek bailout packages has indeed had the effect of large shifts in the VIX seen around the world.

The remainder of the paper is organized as follows. In Section 2 we describe the model. Section 3 considers the model when there is one state variable that is a sunspot, following either an autoregressive or Markov process. Section 4 considers the case where the state variable affects the dividend of the risky asset, thus becoming a macro fundamental. This gives rise to the possibility of sunspot-like equilibria. It then extends the model by introducing a second state variable, which is a sunspot. This allows for equilibria that have the flavor of a switch between fundamental and sunspot-like equilibria. Section 5 concludes.

2 A Simple Mean-Variance Portfolio Choice Model

The model is designed to keep complexity to a strict minimum. Consider an overlapping generation setup where investors are born with wealth W. They live for two periods and only consume when old. Their only problem is to allocate their wealth between a risky equity and a riskfree bond that pays a gross return $R > 1$.

Equity consists of a claim on a tree with a stochastic payoff. There are K trees, each producing an exogenous stochastic output (dividend) A_t. Denoting the equity price by Q_t, the equity return from t to $t+1$ is:

$$R_{K,t+1} = \frac{A_{t+1} + Q_{t+1}}{Q_t} \tag{1}$$

In general, agents face uncertainty both about the dividend and the future equity price. The dividend is equal to :

$$A_t = \bar{A} + mS_t \tag{2}$$

where S_t is an exogenous state variable that follows a stochastic process. The dividend is constant at $\bar{A}$ when $m = 0$, in which case S_t is an extrinsic variable, or a pure sunspot, with no fundamental role. When $m > 0$, S_t has a fundamental impact on the dividend. For simplicity we assume that the distribution of S_{t+1} depends at most on S_t and is time invariant. The analysis could easily be extended to processes of S_{t+1} that depend on more lags of S_t. We also assume that the unconditional distribution of S_t is such that A_t is always non-negative.

Investors born at time t maximize a mean-variance utility over their portfolio return:

$$E_t R^p_{t+1} - 0.5\gamma var_t(R^p_{t+1}) \tag{3}$$

where γ measures risk aversion and the portfolio return is:

$$R^p_{t+1} = \alpha_t R_{K,t+1} + (1 - \alpha_t)R$$

α_t denotes the portfolio share invested in equity. The clearing of the equity market requires that the wealth invested in equity equates the value of existing trees:

$$\alpha_t W = Q_t K \tag{4}$$

Definition 1 *An equilibrium is a non-negative asset price function $Q_t = f(S_t)$ such that (i) agents choose the portfolio share α_t to maximize their utility (3), (ii) the market clearing condition (4) is satisfied and (iii) there are no asset price bubbles: $lim_{T\to\infty} E_t Q_{t+T}/R^{t+T} = 0$.*

Maximization of (3) with respect to α_t gives the optimal portfolio share, which reflects the expected excess return on equity scaled by the variance of the equity return:

$$\alpha_t = \frac{E_t R_{K,t+1} - R}{\gamma var_t(R_{K,t+1})} \tag{5}$$

Using (5), the market clearing condition (4) becomes:

$$E_t(A_{t+1} + Q_{t+1} - RQ_t) = \lambda var_t(Q_{t+1} + A_{t+1}) \tag{6}$$

where $\lambda = \gamma K/W$. Equation (6) equates the equilibrium expected excess payoff on equity to a risk premium that depends on the variance of $Q_{t+1} + A_{t+1}$.

Iterating (6) forward, and using the no bubble condition $lim_{T\to\infty} E_t Q_{t+T}/R^{t+T} = 0$, gives a present value expression for the equilibrium asset price:

$$Q_t = \sum_{i=1}^{\infty} \frac{1}{R^i} E_t A_{t+i} - \lambda \sum_{i=1}^{\infty} \frac{1}{R^i} E_t var_{t+i-1}(Q_{t+i} + A_{t+i}) \tag{7}$$

The asset price depends on the present value of the expected future dividends and the present value of expected future risk, measured by the expected value of $var_{t+i-1}(Q_{t+i} + A_{t+i})$ for $i \geq 1$.

3 Sunspot Equilibria

We first consider the case where the state variable S_t is a pure sunspot with no direct impact on the dividend. This corresponds to $m = 0$ in (2) with the dividend constant at $\bar{A}$. One solution to the asset price is immediate, which is a straightforward *fundamental equilibrium*:

$$Q_t = \bar{A}/(R-1) \tag{8}$$

The asset price is then constant and equal to the present value of the constant dividend.

However, there can be other equilibria where the asset price is affected by the sunspot variable S_t. These are *sunspot equilibria.* In the remainder of this section we focus on these equilibria. Their existence is not guaranteed and depends in particular on the process of S_t. We first derive some necessary conditions for the existence of a sunspot equilibrium, and then consider two specific examples based on respectively an autoregressive and Markov process for S_t.

3.1 Necessary Conditions for Sunspot Equilibrium

When $m = 0$, the present value relationship (7) becomes

$$Q_t = \frac{\bar{A}}{R-1} - \lambda \sum_{i=1}^{\infty} \frac{1}{R^i} E_t var_{t+i-1}(Q_{t+i}) \tag{9}$$

The asset price Q_t can only depend on the sunspot S_t through the asset price risk $E_t var_{t+i-1}(Q_{t+i})$. If risk does not depend on the sunspot, and is therefore constant, it is immediate from (9) that risk must be zero and the fundamental equilibrium (8) is the only solution. A sunspot equilibrium can only exist when the sunspot affects asset price risk. This implies a restriction on the exogenous process for S_t given in the following proposition.[6]

Proposition 1 *A necessary condition for a sunspot equilibrium to exist is that there is at least one pair* $(n_1, n_2) \in N^* \times N^*$ *such that* $cov_t(S_{t+1}^{n_1}, S_{t+1}^{n_2})$ *depends on* S_t*.*

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[6] The notation N^* in the proposition stands for the set of positive natural numbers.

Proof. First, we assumed that the distribution of S_{t+1} depends at most on S_t and is not time varying, Hence, from (9) there can only be a sunspot equilibrium if $var_t(Q_{t+1})$ depends on S_t. Any solution $Q_{t+1} = f(S_{t+1})$ can be written as an infinite order polynominal $Q_{t+1} = \sum_{n=0}^{\infty} \alpha_n S_{t+1}^n$. This implies $var_t(Q_{t+1}) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \alpha_{n_1} \alpha_{n_2} cov_t(S_{t+1}^{n_1}, S_{t+1}^{n_2})$. It therefore follows that for $var_t(Q_{t+1})$ to depend on S_t there must be at least one pair (n_1, n_2) with $n_1 \geq 1$ and $n_2 \geq 1$ such that $cov_t(S_{t+1}^{n_1}, S_{t+1}^{n_2})$ depends on S_t. ■

While Proposition 1 is not very intuitive, an obvious condition for Proposition 1 to hold is that the distribution of S_{t+1} depends on S_t. This can be stated in the following Proposition.

Proposition 2 *A necessary condition for a sunspot equilibrium to exist is that the distribution of* S_{t+1} *depends on* S_t.

If this condition does not hold, it breaks a key element in the circular link between the asset price and asset price risk that gives rise to the possibility of sunspot equilibria. If asset price risk depends on S_t, so will the asset price Q_t. Therefore Q_{t+1} depends on S_{t+1}. But when the distribution of S_{t+1} does not depend on S_t it is immediate that asset price risk $var_t(Q_{t+1})$ will not depend on S_t as assumed, so that there cannot be sunspot equilibria.

The condition in Propostion 2 is looser than in Proposition 1. To see this, consider the case where S_t follows a symmetric Markov process. It can take two values, $\bar{S}$ and $\underline{S}$, and the probability of remaining in the same state is $p > 0.5$. This implies that the distribution of S_{t+1} depends on S_t, so that the condition in Proposition 2 is satisfied. However, $cov_t(S_{t+1}^{n_1}, S_{t+1}^{n_2}) = p(1-p)(\bar{S}^{n_1} - \underline{S}^{n_1})(\bar{S}^{n_2} - \underline{S}^{n_2})$ is independent of the value of S_t. The condition of Proposition 1 is therefore not satisfied and there cannot be a sunspot equilibrium. This is because both the probability and the absolute size of a jump to another state do not depend on the current state.

While the conditions in Propositions 1 and 2 are satisfied for a wide range of distributions, in the remainder of this section we focus on two examples to illustrate the existence of sunspot equilibria. We first consider an AR(1) process before turning to an asymmetric two-state Markov process.

3.2 Autoregressive Sunspot Process

We assume that the sunspot follows an AR(1) process:

$$S_{t+1} = \rho S_t + \epsilon_{t+1} \qquad \text{with } 0 < \rho < 1 \tag{10}$$

The innovation ϵ_{t+1} has a bounded zero-mean symmetric distribution with $\epsilon_{t+1} \in [-\bar{\epsilon}, \bar{\epsilon}]$. The variance of ϵ_{t+1} is denoted by σ^2, and the variance of ϵ_{t+1}^2 is denoted ω^2. Clearly the condition in Proposition 2 is satisfied when $\rho > 0$. The condition in Proposition 1 is satisfied as well because $cov(S_{t+1}^2, S_{t+1}^2) = 4\rho^2\sigma^2 S_t^2 + \omega^2$ depends on S_t.

For the purpose of the following Proposition it is convenient to define a threshold value on the dividend that insures a non-negative asset price:

$$A_1 \equiv \frac{R-\rho^2}{4\lambda\rho^2\sigma^2}\left(\frac{R-\rho^2}{4\rho^2\sigma^2}\omega^2 + \sigma^2 + \frac{R-1}{(1-\rho)^2}\bar{\epsilon}^2\right)$$

The following Proposition shows that there exists a sunspot equilibrium.

Proposition 3 *Assume that the sunspot variable S_t follows the AR process (10) and that $\bar{A} > A_1$. Then there exist two equilibria in the class of finite polynomial solutions: the fundamental equilibrium (8) and a sunspot equilibrium*

$$Q_t = \tilde{Q} - VS_t^2 \tag{11}$$

where

$$V = \frac{R-\rho^2}{4\lambda\rho^2\sigma^2} > 0 \tag{12}$$

$$\tilde{Q} = \frac{1}{R-1}\left(\bar{A} - \lambda V^2\omega^2 - V\sigma^2\right) < \frac{\bar{A}}{R-1} \tag{13}$$

Proof: see Appendix.

To understand the existence of the sunspot equilibrium it is useful to go back to the present value relation (9). In the sunspot equilibrium, risk is time-varying in the sunspot variable:

$$var_t(Q_{t+1}) = 4V^2\rho^2\sigma^2 S_t^2 + V^2\omega^2$$

There are therefore self-fulfilling shifts in perceptions of risk. This is an equilibrium because of the circular relationship between the stochastic process of the asset price and asset price risk. If agents perceive asset price risk to depend quadratically on the sunspot, then so does Q_t from (9). Risk then depends on the variance of S_{t+1}^2, which depends on S_t^2 when $\rho > 0$. Beliefs about risk are therefore self-fulfilling and coordinated around the sunspot variable. The asset price is lower in the sunspot equilibrium than in the fundamental equilibrium and risk is higher and more volatile.

We should finally say that while for a particular AR(1) process we have identified only one sunspot equilibrium, within the class of AR(1) processes there is an infinite number of such equilibria as it applies for any $\rho > 0$ and an infinite number of values of σ, ω and $\bar{\epsilon}$. Also, even for a given set of parameters we cannot rule out that there are additional sunspot equilibria within the class of infinite order polynomials.

3.3 Two-State Markov Sunspot Process

We next consider the example of a two-state asymmetric Markov process. We refer to the two states as the "normal" and the "bad" state, denoted by N and B respectively. We denote the probability of being in a state $i = N, B$ next period, conditional on being in that state today, by p_i. We assume that p_B and p_N are both between 0.5 and 1 and that the normal state is more persistent: $p_N > p_B$.[7] Of course, what agents consider as "bad" is subjective as S_t plays no fundamental role when $m = 0$.

Define $p_D = p_B(1-p_B) - p_N(1-p_N)$ and $\kappa = 1 + R - p_N - p_B$, which are both positive under our assumptions. We also define the following threshold value for the dividend:

$$A_2 \equiv \frac{1-p_B}{p_D}\frac{\kappa}{\lambda}\left[\kappa\frac{p_B}{p_D} - 1\right]$$

The value of the asset price in state i is denoted by Q_i. The following Proposition shows that there is exactly one sunspot equilibrium in this case.

Proposition 4 *Assume that the sunspot variable S_t follows a 2-state Markov process with transition probabilities p_i, $i = N, B$, of staying in state i. Assume*

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[7] If $p_N = p_B$, S_t follows a symmetric two-state Markov process. We have already seen that this does not satisfy the necessary condition for the existence of a sunspot equilibrium in Proposition 1.

that $0.5 < p_B < p_N < 1$ *and* $\bar{A} > A_2$*. Then there are two equilibria. One is the fundamental equilibrium where* $Q = \bar{A}/(R-1)$ *and there is no risk. The second is a sunspot equilibrium where the price is always lower than in the fundamental equilibrium and we have*

$$Q_D = Q_N - Q_B = \frac{\kappa}{\lambda p_D} > 0 \tag{14}$$

and

$$Q_B = \frac{1}{R-1}\left(\bar{A} - \lambda p_B(1-p_B)Q_D^2 + (1-p_B)Q_D\right) > 0 \tag{15}$$

Proof: see Appendix.

As was the case for an autoregressive sunspot process, the asset price in the sunspot equilibrium is always lower than in the fundamental equilibrium, while risk is positive and varies across the two states. In state i the variance of Q_{t+1} is $p_i(1-p_i)Q_D^2$. Since $p_B(1-p_B) > p_N(1-p_N)$ under our assumptions, risk is higher in the bad state, which results in a lower price in the bad state.

The intuition for the sunspot equilibrium again reflects the circular relationship between the stochastic process of the asset price and asset price risk. If agents believe that asset price risk is high in state B and low in state N, then indeed it will be. It leads to a low price in state B and a high price in state N. This in turn implies that risk is higher in state B as $p_B < p_N$ means that in state B there is more uncertainty about next period's state and therefore about next period's price. Shifts in beliefs about risk across the two states are therefore self-fulfilling.

When the increase in risk from state N to state B is very large, and the drop in the price big, we can speak of a risk panic. This is a large self-fulfilling shift in perceived risk. To illustrate this, Figure 1 shows both the asset price (left panel) and asset price risk (right panel) for a particular parameterization. Asset price risk is defined as the standard deviation of next period's asset price divided by the asset price today. It is assumed that $p_N = 0.99$, so that a switch to the bad state is quite rare. The solution is shown for different values of p_B. Independent of the value of p_B we see that a switch from state N to state B leads to an enormous spike in risk and drop in the price level.[8] For example, for $p_B = 0.7$ the risk panic involves an increase in asset price risk from 5% to 40% and a drop in the asset price by 47%.

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[8] If p_B is too high (even if still below p_N) there is no sunspot equilibrium as the condition $\bar{A} > A_2$ is no longer satisfied.

4 Sunspot-Like Equilibria and Risk-Panics

We now turn to the case where $m > 0$, so that the state variable S_t is a fundamental that affects the dividend. We show that apart from a fundamental equilibrium there now exist so-called "sunspot-like" equilibria. In those equilibria S_t plays the dual role of a fundamental that affects the asset price through its impact on dividends and a sunspot that leads to self-fulfilling shifts in risk. We again consider the cases where S_t follows a first-order autoregressive process and an asymmetric two-state Markov process. We also consider an extension of the model where in addition to the time-varying fundamental (the dividend) there is a sunspot variable that can trigger risk panics.

4.1 Autoregressive Dividend Process

We assume that the process for S_t is given by (10) with the same assumptions as before. In addition we define the following threshold values for the dividend: $A_3 \equiv max(A_{31}, A_{32}, A_{33})$, where:

$$\begin{aligned}
A_{31} &\equiv m\bar{\epsilon}/(1-\rho) \\
A_{32} &\equiv \lambda\frac{R^2m^2\sigma^2}{(R-\rho)^2} + \frac{(R-1)m\rho\bar{\epsilon}}{(R-\rho)(1-\rho)} \\
A_{33} &\equiv \frac{R-\rho^2}{4\lambda\rho^2\sigma^2}\left(\sigma^2 + \frac{(R-1)\bar{\epsilon}^2}{(1-\rho)^2} + \frac{(R-\rho^2)\omega^2}{4\rho^2\sigma^2}\right) + \lambda\frac{\rho^2m^2\sigma^2}{(1-\rho)^2} + \frac{(R-1)m\bar{\epsilon}}{(1-\rho)^2}
\end{aligned}$$

Proposition 5 *Assume that S_t follows the AR process (10). Also assume that $\bar{A} > A_3$. Then there exist two equilibria in the class of finite polynomials. The first is a fundamental equilibrium:*

$$Q_t = \frac{1}{R-1}\left(\bar{A} - \lambda\frac{R^2m^2\sigma^2}{(R-\rho)^2}\right) + \frac{m\rho}{R-\rho}S_t \tag{16}$$

The second is a sunspot-like equilibrium

$$Q_t = \tilde{Q} + vS_t - VS_t^2 \tag{17}$$

where

$$V = \frac{R-\rho^2}{4\lambda\rho^2\sigma^2} \tag{18}$$

$$v = -\frac{m}{1-\rho} \tag{19}$$

$$\tilde{Q} = \frac{1}{R-1}\left(\bar{A} - \lambda(V^2\omega^2 + (v+m)^2\sigma^2) - V\sigma^2\right) \tag{20}$$

Proof: see Appendix.

In this case, Q_t is obviously affected by S_t even in the fundamental equilibrium. To see the contrast between the fundamental and sunspot-like equilibrium, consider the present value equation (7), which in this case becomes

$$Q_t = \frac{1}{R-1}\bar{A} + \frac{m\rho}{R-\rho}S_t - \lambda\sum_{i=1}^{\infty}\frac{1}{R^i}E_t var_{t+i-1}(Q_{t+i} + A_{t+i}) \tag{21}$$

In the fundamental equilibrium the asset payoff risk $var_t(Q_{t+1}+A_{t+1}) = (mR\sigma)^2/(R-\rho)^2$ is constant, so that the last term in (21) is constant. The asset price then depends positively on S_t with coefficient $m\rho/(R-\rho)$. It is more sensitive to dividend shocks when the dividend is more peristent (higher ρ). The impact of S_t on the asset price vanishes to zero as its fundamental impact becomes small ($m \rightarrow 0$).

In the sunspot-like equilibrium, S_t plays the dual role of a fundamental and a sunspot that leads to time-varying beliefs about risk. Its fundamental role is still captured by the second term of (21) that depends positively on S_t. Its sunspot role is captured by the present value of time-varying risk through the last term of (21). We have

$$var_t(Q_{t+1} + A_{t+1}) = (v + m - 2\rho V S_t)^2\sigma^2 + V^2\omega^2 \tag{22}$$

This time-varying risk is self-fulfilling as it does not go away when the fundamental role of S_t vanishes with $m \rightarrow 0$. This is because V does not depend on m. The coefficient on the quadratic term is in fact the same as in the pure sunspot equilibrium. When $m \rightarrow 0$ the sunspot-like equilibrium converges to the pure sunspot equilibrium in Proposition 3. The main difference with Section 3 is that when $m > 0$ the self-fulfilling shifts in beliefs about risk are now coordinated around a macro fundamental rather than an external sunspot variable.

Note also that the linear coefficient on S_t is negative in the sunspot-like equilibrium. This is because the positive linear term in S_t in (21), associated with its fundamental role, is more than offset by the linear dependence of risk (22) on S_t that captures self-fulfilling beliefs about risk. Together with the negative quadratic term in S_t, it is therefore clear that the sunspot role dominates the fundamental role in the sunspot-like equilibrium.

Although in a very different context, not involving time-varying shifts in risk, Manuelli and Peck (1992) and Spear, Srivastava and Woodford (1990) also present models with sunspot-like equilibria. Spears, Srivastava and Woodford (1990) point out that "...a sharp distinction between "sunspot equilibria" and "non-sunspot equilibria" is of little interest in the case of economies subject to stochastic shocks to fundamentals." Indeed, as we raise m slightly above 0, the sunspot-like equilibrium is technically no longer a pure sunspot equilibrium, but it is effectively indistinguishable.

There is one unattractive aspect of the sunspot-like equilibrium. This relates to the quadratic relation between risk and S_t:

$$var_t(Q_{t+1}) = (v - 2\rho V S_t)^2\sigma^2 + V^2\omega^2 \tag{23}$$

Since $v < 0$, this implies that the equilibrium has the odd property that risk is highest, and the asset price is lowest, when the fundamental is strongest. This happens when S_t reaches its maximum value of $\bar{\epsilon}/(1-\rho)$, where the dividend is the highest. Related to this, a rise in dividends always lowers the asset price when $S_t > 0$. This unappealing result is not a general feature of sunspot-like equilibria, but instead reflects the particular process for S_t considered here. It does not occur when S_t follows an asymmetric two-state Markov process, which we turn to next.

4.2 Two-State Markov Dividend Process

The asymmetric two-state Markov process is analogous to that in section 3.3, with the difference that the dividend is higher in the normal state than in the bad state: $A_N > A_B$. We define $A_D = A_N - A_B$, which converges to zero when $m \to 0$. The assumptions on the switching probabilities are as in section 3.3, and we define the following threshold for the dividend:

$$A_4 = \frac{1-p_B}{p_D}\left(-p_B R A_D + \frac{p_B\left(R-p_N\right)+p_N(1-p_N)}{2\lambda p_D}\left[\kappa + \sqrt{\kappa^2 - 4R\lambda p_D A_D}\right]\right)$$

The equilibria are then given by the following proposition.

Proposition 6 *Assume that the fundamental A_t follows a 2-state Markov process. It takes on value A_i in state $i = N, B$ with transition probability p_i of remaining* *in state i. Assume that* $0 < p_B < p_N < 1$, $A_D < \frac{\kappa^2}{4R\lambda p_D}$ *and* $A_B > A_4$*. Then there are two equilibria. The values of the asset price difference* $Q_D = Q_N - Q_B$ *in the two equilibria are:*

$$Q_D = \left[\frac{\kappa}{2\lambda p_D} - A_D\right] \pm \frac{1}{2\lambda p_D}\left[\kappa^2 - 4R\lambda p_D A_D\right]^{0.5} > 0 \tag{24}$$

Corresponding to each value of Q_D*, the asset price in state B is*

$$Q_B = \frac{A_B}{R-1} + \frac{(1-p_B)p_N(p_N+p_B-1)A_D - (1-p_B)[p_B(R-p_N)+p_N(1-p_N)]Q_D}{p_D(R-1)} \tag{25}$$

Proof: see Appendix.

Since Q_D is positive, the asset price is higher in the normal state than in the bad state. We refer to the equilibrium where Q_D takes on its lowest value (with the minus sign between the two terms in (24)) as the fundamental equilibrium, and to the other as the sunspot-like equilibrium.

In the fundamental equilibrium, the asset price differs between the normal and bad states only if the dividend differs between these two states ($Q_D \to 0$ when $A_D \to 0$). By contrast, the asset price differs across the two states in the sunspot-like equilibrium even when the dividend does not. When A_D goes to zero, Q_D converges to $\kappa\left(\lambda p_D\right)^{-1}$, which is its value in the pure sunspot equilibrium (14) in Section 3.3.

In the fundamental equilibrium asset payoff risk in state $i = N, B$ is written as

$$var_t(Q_{t+1} + A_{t+1}) = p_i(1-p_i)\left(\frac{1}{2\lambda p_D}\right)^2\left[\kappa - \left[\kappa^2 - 4R\lambda p_D A_D\right]^{0.5}\right]^2 \tag{26}$$

Equation (26) shows that risk is higher in the bad state as $p_B(1-p_B) > p_N(1-p_N)$, which reflects our assumption that the fundamental is riskier (i.e. more likely to change) in the bad state.[9] Risk goes to zero when the dividend becomes identical in the two states ($A_D \to 0$). The higher price in the normal state in the fundamental equilibrium follows both from the higher expected dividend in the normal state and the lower risk.

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[9] The variance of A_{t+1} is $p_i(1-p_i)A_D^2$ in state i.

Things are quite different in the sunspot-like equilibrium. Asset payoff risk in state i is now written as

$$var_t(Q_{t+1} + A_{t+1}) = p_i(1 - p_i)\left(\frac{1}{2\lambda p_D}\right)^2 \left[\kappa + \left[\kappa^2 - 4R\lambda p_D A_D\right]^{0.5}\right]^2 \tag{27}$$

It is still the case that risk is higher in the bad state than in the good state. This is however no longer because of the exogenously higher fundamental risk in the bad state. To the contrary, (27) shows that asset payoff risk (27) increases in both states when fundamental risk declines ($A_D \to 0$). In the sunspot-like equilibrium risk is self-fulfilling and the main role of the fundamental S_t is as a focal point for these self-fulfilling shifts in risk. Also note that the sunspot-like equilibrium does not have the odd features that arise when S_t follows an AR process. A higher fundamental now unambiguously implies a higher price and lower risk.

4.3 Switching Equilibria and Risk Panics

As a final exercise we consider a situation that combines elements of both the sunspot and sunspot-like equilibria considered so far. In addition to the Markov process for the fundamental between states B and N in the previous subsection, we extend the model by introducing a two-state sunspot variable, with the states indexed as 1 and 2. The dividend is not affected by whether we are in state 1 or 2.

The presence of two state variables allows a richer analysis of risk panics. It is useful to think of the sunspot in this case as a trigger variable that shifts expectations between low risk in state 1 and high risk in state 2. One can think of state 2 as a “panic state”, so that the switch to that state implies a large spike in risk and drop in the asset price. We will show that while this panic is not caused by a change in the fundamental (the dividend), the magnitude of the panic depends critically on the level of the fundamental. This feature is absent from the equilibria considered thus far.

The sunspot state variable is assumed to be uncorrelated with the fundamental. In either state 1 or 2, the probability of remaining in the same state next period is $p > 0.5$. The model now has four possible states, depending on the value of the sunspot and the fundamental: $(N, 1)$, $(N, 2)$, $(B, 1)$ and $(B, 2)$. We define the asset prices in these states as respectively $Q_N(1)$, $Q_N(2)$, $Q_B(1)$ and $Q_B(2)$. We solve the asset prices in each state by imposing a market clearing condition for each state. We also define $Q_D(i) = Q_N(i) - Q_B(i)$ for $i = 1, 2$.

In this case the equilibria involve longer expressions that relate the asset price in each state to model parameters, which are fully described in the Appendix. Instead, the proposition below focuses on some key signs that characterize the new equilibrium that results from this setting. In the Appendix we define a cut off A_5 for A_B and A^{max} for A_D. There is also a critical value for p :

$$\bar{p} = \frac{3R + 1 - p_N - p_B}{4R + 2 - 2p_N - 2p_B}$$

which is between 0.75 and 1.

Proposition 7 *Assume that the fundamental* A_t *follows a 2-state Markov process as in Proposition 6. Also assume that* $A_D < A^{\max}$ *and that* $A_B > A_5$*. Then there are 4 equilibria when* $\bar{p} < p < 1$*. The first two equilibria are the same as the fundamental and sunspot-like equilibria in Proposition 6, regardless of whether we are in state 1 or 2. In the third equilibrium we have:*

$$Q_D(2) > Q_D(1) > 0 \tag{28}$$

$$Q_B(2) - Q_B(1) = \delta(Q_D(1) - Q_D(2)) < 0 \tag{29}$$

$$Q_N(2) - Q_N(1) = (\delta - 1)(Q_D(1) - Q_D(2)) < 0 \tag{30}$$

with $\delta > 1$*. Equilibrium 4 is analogous to equilibrium 3, with the role of states 1 and 2 switched.*

Proof: see Appendix.

Equilibrium 3 is the novel result in Proposition 7.[10] A switch from state 1 to 2 involves an increase in risk and a drop in the asset price. Proposition 7 states that asset prices are lower in state 2: $Q_B(2) < Q_B(1)$ and $Q_N(2) < Q_N(1)$.This is associated with an increase in risk as there is no change in the expected level of the dividend. A numerical illustration below shows that the increase in risk and drop in the price can be very large, in which case we can speak of a risk panic.

We show in the Appendix that when p approaches 1 equilibrium 3 is such that state 1 converges to the fundamental equilibrium in Proposition 6, while

---

[10] Equilibrium 4 is basically the same as Equilibrium 3.

state 2 converges to the sunspot-like equilibrium. A switch from state 1 to state 2 then implies a switch from the fundamental to the sunspot-like equilibrium of Proposition 6. When $p < 1$ a switch from state 1 to state 2 is not exactly a switch from two equilibria of Proposition 6, as the very possibility of a switch increases uncertainty when we are in state 1. State 1 is then characterized by higher risk and a lower asset price than in the fundamental equilibrium in Proposition 6.

Three features characterize the role of the fundamental during and after a risk panic. First, the panic itself is not caused by a change in the fundamental, as it consists of a switch between state 1 and state 2, and not between state N and state B. Second, the panic has a larger impact when the fundamental is weak to start with. It follows from Proposition 7 that $Q_B(2) - Q_B(1) < Q_N(2) - Q_N(1)$. The drop in the asset price in a panic is larger when the fundamental is bad (state B) than when it is normal (state N). Finally, after a panic (once we are in state 2) the asset price becomes more volatile. As $Q_D(2) > Q_D(1)$, the asset price is more sensitive to changes in the fundamental between states N and B when we are in state 2.

Even though the panic is not caused by a change in the fundamental, the last two results show that the fundamental plays a key role as a focal point for expectations that affects both the magnitude of the panic itself, and subsequent shifts in perceived risk. When p is close to 1 we can think of the role of S_t as suddenly changing from that of a pure fundamental to that of a sunspot-like variable around with agents coordinate their perceptions of risk.

Figure 2 provides a numerical illustration of these results for a particular parameterization. The probability of staying in the normal state is $p_N = 0.99$, while the probability of staying in the bad state is $p_B = 0.9$. This means that the fundamental is 91% of the time in the normal state and 9% of the time in the bad state. The probability that the sunspot variable remains the same is $p = 0.99$. If we are currently in the low-risk state 1, the probability of switching to panic state 2 is then only 0.01.

Figure 2 considers the following experiment. We start in state $(N, 1)$ in periods 0 and 1, where the dividend is at its normal value of $A_N = 1$ and we are in the non-panic state 1. Then at time 2 the dividend drops by 10% to its value $A_B = 0.9$, but we remain in the non-panic state 1. In period 3 we switch to the panic state 2 while the fundamental remains weak at $A_B = 0.9$. In period 4 the fundamental is restored to its normal level of $A_N = 1$ but we remain in the panic state 2. Finally, starting with period 5 we return to state $(N, 1)$. Figure 2 reports both the asset price (normalized to 100 in state $(N, 1)$) and asset price risk. The latter is the standard deviation of the asset price next period, divided by the asset price today.

We see that the deterioration of the fundamental in period 2 lowers the asset price by about 13%. About a third of that is a result of the lower expect future dividend while the rest is the result of the exogenous increase in risk. We see though that risk spikes much more in period 3 when the economy is hit by a risk panic (a switch to state 2). This causes a much sharper additional drop in the asset price, lowering it to a level 56% below its starting point. What is key for this really bad outcome is that both the fundamental is weak (we are in state B) and the economy is hit by a self-fulfilling risk panic. In period 4, when the fundamental is restored to its normal level, the asset price is way up again (only 10% below its starting point), even though we are still in the panic state.

Figure 2 therefore illustrates that the level of the fundamental plays a key role during a risk panic. The panic is much larger when the fundamental is weak at the time of the panic. Moreover, once we reach the panic state, the asset price becomes much more sensitive to changes in the fundamental. An improvement in the fundamental from state B to state N raises the asset price much more when we are in the panic state (compare period 4 to period 3) than when we are in the non-panic state (compare period 1 to period 2). Rather than a regular fundamental, S_t becomes a gauge of fear when we switch to the panic state.

5 Conclusion

We have developed a very simple mean-variance portfolio choice model to show that self-fulfilling shifts in risk, coordinated around either a sunspot or a macro fundamental, can occur in equilibrium. This is a result of a circular relationship between the process of asset price risk and the asset price itself. The analysis was motivated by large changes in asset price risk during recent financial crises. We have shown that the model can give rise to significant risk panics that take the form of a very large sudden spike in risk and drop in the asset price. The magnitude of such panics can be particularly large when a macro fundamental is weak. This has indeed been the case during recent crises (weak financial institutions, large debt).

The analysis raises various questions that deserve to be addressed in future research. First, one can ask what happens when we consider multiple risky assets. For example, will all asset prices be equally affected by the panic? This question is particularly relevant when considering the global dimension of recent financial crises, with stock markets usually changing in lockstep around the world. In Bacchetta and van Wincoop (2011) we have started to address this question. Second, what are the policy implications? Is there anything that regulators can do to reduce the magnitude of risk panics? In Bacchetta, Tille and van Wincoop (2011) we argue that a policy aimed at penalizing balance sheet risk exposure of financial institutions can help. Finally, it would be of interest to enrich the model to introduce other features often seen during financial panics, such as a drop in market liquidity, banks runs and significant contractions of the real economy.

Appendix

A Proofs

A.1 Proposition 3

First, conjecture that the solution is $Q_{t+1} = \tilde{Q} - VS_{t+1}^2$. Using (10), we have $Q_{t+1} = \tilde{Q} - V\rho^2 S_t^2 - 2V\rho S_t \epsilon_{t+1} - V\epsilon_{t+1}^2$. The expectation and variance of Q_{t+1} are therefore

$$
\begin{aligned}
E_t Q_{t+1} &= \tilde{Q} - V\rho^2 S_t^2 - V\sigma^2 \\
var_t(Q_{t+1}) &= 4V^2\rho^2\sigma^2 S_t^2 + V^2\omega^2
\end{aligned}
$$

Notice that we used the fact that $E_t\epsilon_{t+1}^3 = 0$ given the symmetry of the distribution. Substituting these into the market clearing condition (6) implies

$$\bar{A} + \tilde{Q} - V\rho^2 S_t^2 - V\sigma^2 - R\tilde{Q} + RVS_t^2 = \lambda\left(4V^2\rho^2\sigma^2 S_t^2 + V^2\omega^2\right)$$

Equating the constant terms on the left and right hand side, as well as the terms proportional to S_t^2, gives

$$
\begin{aligned}
\bar{A} + \tilde{Q}(1-R) - V\sigma^2 &= \lambda V^2\omega^2 \\
V(R-\rho^2) &= \lambda 4V^2\rho^2\sigma^2
\end{aligned}
$$

This has two solutions. One is the fundamental equilibrium where $V = 0$ and $\tilde{Q} = \bar{A}/(R-1)$. The other is the sunspot equilibrium where V and $\tilde{Q}$ are as in (12) and (13). The condition $\bar{A} > A_1^{\min}$ implies that Q_t is non-negative. The lowest value of the asset price is reached when ϵ_t is constant at $\bar{\epsilon}$ or $-\bar{\epsilon}$. In that case $(S_t)^2$ reaches its maximum value of $\bar{\epsilon}^2/(1-\rho)^2$ and Q_t its lowest value of $\tilde{Q} - V\bar{\epsilon}^2/(1-\rho)^2$. Substituting the values for $\tilde{Q}$ and V, this is positive when $\bar{A} > A_1^{\min}$. Finally, it is clear that the solution satisfies the no bubble condition $lim_{T\to\infty} E_t(1/R)^T Q_{t+T} = 0$ as Q_t is always between 0 and $\bar{A}/(R-1)$.

Moreover, the sunspot equilibrium (11) is the only one within the class of finite polynomial functions. Assume that the solution is

$$Q_t = \sum_{i=1}^{n} \alpha_i S_t^i \tag{31}$$

so that $Q_{t+1} = \sum_{i=1}^{n} \alpha_i (\rho S_t + \epsilon_{t+1})^i$. When taking both the expectation and the variance of this expression the term with highest power of S_t is the variance of $\alpha_n n \rho^{n-1} S_t^{n-1} \epsilon_{t+1}$, which is

$$\alpha_n^2 n^2 \rho^{2(n-1)} \sigma^2 S_t^{2(n-1)}$$

Hence, there is a term in the market-clearing condition with the power $2(n-1)$. Considering (31), this implies that $\alpha_n = 0$ for all $n > 2$.

A.2 Proposition 4

If we are in state N at time t, then

$$E_t Q_{t+1} = p_N Q_N + (1 - p_N) Q_B \tag{32}$$

$$var_t(Q_{t+1}) = p_N (1 - p_N)(Q_N - Q_B)^2 \tag{33}$$

Similarly, if we are in state B at time t, then

$$E_t Q_{t+1} = p_B Q_B + (1 - p_B) Q_N \tag{34}$$

$$var_t(Q_{t+1}) = p_B (1 - p_B)(Q_N - Q_B)^2 \tag{35}$$

Substituting these results into (6), the market clearing conditions in respectively states N and B can we written as

$$\bar{A} + p_N Q_N + (1 - p_N) Q_B - R Q_N = \lambda p_N (1 - p_N)(Q_N - Q_B)^2 \tag{36}$$

$$\bar{A} + p_B Q_B + (1 - p_B) Q_N - R Q_B = \lambda p_B (1 - p_B)(Q_N - Q_B)^2 \tag{37}$$

Taking the difference between these two relations, we have

$$\kappa Q_D = \lambda p_D Q_D^2 \tag{38}$$

This has two solutions. The first is $Q_D = 0$, which gives the fundamental equilibrium $Q_N = Q_B = \bar{A}/(R-1)$. The second solution is the sunspot equilibrium where $Q_D = \kappa / (\lambda p_D)$. Substituting this into the market clearing condition for state B, using that $Q_N = Q_B + Q_D$, we get (15). Using the expression for Q_D, Q_B is then positive when $\bar{A} > A_2^{\min}$. Since Q_N is larger than Q_B, it is positive as well. The no-bubble condition is clearly satisfied as the asset price can take on only two finite values.

We can also show that the asset price is always higher than in the fundamental equilibrium. For this it is sufficient to show that $Q_N > barA/(R-1)$. Using (15) and $Q_N = Q_B + Q_D$, this condition is satisfied when $-\lambda p_B(1-p_B)Q_D + (R - p_B) < 0$. After substituting the expression for Q_D, the left hand side becomes $-(1-p_N)\left(p_B(1-p_B) + (R-p_N)p_N\right)/p_D$, which is indeed negative.

A.3 Proposition 5

First, conjecture the solution $Q_t = \tilde{Q} + vS_t - VS_t^2$. From the process (10), we have $Q_{t+1} + A_{t+1} = \tilde{Q} + \bar{A} + (v+m)\rho S_t - V\rho^2 S_t^2 + (v + m - 2V\rho S_t)\epsilon_{t+1} - V\epsilon_{t+1}^2$. The expectation and variance of $Q_{t+1} + A_{t+1}$ are therefore:

$$\begin{aligned} E_t(Q_{t+1} + A_{t+1}) &= \tilde{Q} + \bar{A} + (v+m)\rho S_t - V\rho^2 S_t^2 - V\sigma^2 \\ var_t(Q_{t+1} + A_{t+1}) &= (v + m - 2V\rho S_t)^2\sigma^2 + V^2\omega^2 \end{aligned}$$

Substituting these into the market clearing condition (6) implies

$$\tilde{Q} + \bar{A} + (v+m)\rho S_t - V\rho^2 S_t^2 - V\sigma^2 - R\tilde{Q} - RvS_t + RVS_t^2 = \lambda\left((v + m - 2\rho VS_t)^2\sigma^2 + V^2\omega^2\right)$$

Equating the terms proportional to S_t^2, S_t and constant terms on the left and right hand side gives

$$\begin{aligned} V(R-\rho^2) &= \lambda 4V^2\rho^2\sigma^2 \\ m\rho + v(\rho - R) &= -\lambda 4(v+m)V\rho\sigma^2 \\ \bar{A} + \tilde{Q}(1-R) - V\sigma^2 &= \lambda\left((v+m)^2\sigma^2 + V^2\omega^2\right) \end{aligned}$$

The first equation implies that either $V = 0$ or $V = \frac{R-\rho^2}{4\lambda\rho^2\sigma^2}$. When $V = 0$ the other two equations imply that $v = \frac{m\rho}{R-\rho}$ and $\tilde{Q} = \frac{1}{R-1}\left(\bar{A} - \lambda\frac{R^2m^2\sigma^2}{(R-\rho)^2}\right)$. This is the fundamental equilibrium. When $V = \frac{R-\rho^2}{4\lambda\rho^2\sigma^2}$, the other two equations imply that v and $\tilde{Q}$ are as in respectively (19) and (20). This is the sunspot-like equilibrium.

The lowest value that the dividend can take is when S_t is at its lowest value of $-\bar{\epsilon}/(1-\rho)$. The dividend is always positive when $\bar{A} > A_{31}$. In the fundamental equilibrium the lowest value that the asset price can take is when S_t is at its lowest value of $-\bar{\epsilon}/(1-\rho)$. The asset price is then positive when $\bar{A} > A_{32}$. In the sunspot-like equilibrium the lowest value that the asset price can take is when S_t is at its highest value of $\bar{\epsilon}/(1-\rho)$. The asset price is then positive when $\bar{A} > A_{33}$. Therefore the condition $\bar{A} > A_3^{\min}$ guarantees that the dividend and the asset prices in both equilibria are always positive.

The no-bubble condition is clearly satisfied as well, as the asset price is bounded in both equilibria because S_t is bounded. Finally, the same reasoning as in Proposition 2 can be used to show that the linear and quadratic solutions are the only ones among the class of finite polynomials.

A.4 Proposition 6

If we are in state N at time t, then

$$E_t(Q_{t+1} + A_{t+1}) = p_N(Q_N + A_N) + (1 - p_N)(Q_B + A_B) \tag{39}$$

$$var_t(Q_{t+1} + A_{t+1}) = p_N(1 - p_N)(A_D + Q_D)^2 \tag{40}$$

Similarly, if we are in state B at time t, then

$$E_t(Q_{t+1} + A_{t+1}) = (1 - p_B)(Q_N + A_N) + p_B(Q_B + A_B) \tag{41}$$

$$var_t(Q_{t+1} + A_{t+1}) = p_B(1 - p_B)(A_D + Q_D)^2 \tag{42}$$

Substituting these results into (6), using $Q_N = Q_B + Q_D$ and $A_N = A_B + A_D$, the market clearing conditions in respectively states N and B can we written as

$$\begin{aligned} p_N\left[A_D + Q_D\right] - RQ_D + A_B - (R-1)\,Q_B &= \lambda p_N\,(1 - p_N)\left[A_D + Q_D\right]^2 \\ (1 - p_B)\left[A_D + Q_D\right] + A_B - (R-1)\,Q_B &= \lambda p_B\,(1 - p_B)\left[A_D + Q_D\right]^2 \end{aligned}$$

Taking the difference, defining $x = A_D + Q_D$, we have

$$\lambda p_D x^2 - \kappa x + RA_D = 0 \tag{43}$$

This quadratic polynomial has two solutions when $A_D < \frac{\kappa^2}{4R\lambda p_D}$. These two solutions are:

$$x = \frac{1}{2\lambda p_D}\left(\kappa \pm \left[\kappa^2 - 4R\lambda p_D A_D\right]^{0.5}\right)$$

Using $Q_D = x - A_D$, this implies (24). The corresponding values of Q_B can be found from the market clearing condition for state B. Replacing $A_D + Q_D$ with x and using $\lambda p_D x^2 = \kappa x - RA_D$, the state B market clearing condition becomes

$$Q_B = \frac{A_B}{R-1} + \frac{1 - p_B}{(R-1)p_D}\left((p_D - p_B\kappa)x + Rp_BA_D\right) \tag{44}$$

This implies (25), using $x = Q_D + A_D$ and $p_D - p_B\kappa = -(p_B(R-p_N)+p_N(1-p_N))$ from the definition of κ.

The only thing that remains to be checked is that the asset price is always positive. First note that Q_D is positive in both equilibria. This follows from the polynomial in x, which implies $x = \frac{\lambda p_D}{\kappa}x^2 + \frac{R}{\kappa}A_D$, so that $Q_D = x - A_D = \frac{\lambda p_D}{\kappa}x^2 + \frac{1}{\kappa}(R-\kappa)A_D$. This is positive because $A_D > 0$ and $R - \kappa = p_N + p_B - 1 > 0$. The asset price is then guaranteed to always be positive when it is positive at the lowest value of Q_B. Substituting the higher of the two roots of x into (44), the resulting expression is larger than 0 when $A_B > A_4^{\min}$.

A.5 Proposition 7

We start from the market clearing condition (6), with one condition for each of the four states $(N,1)$, $(N,2)$, $(B,1)$ and $(B,2)$. We simplify this to a system of three equations in three unknowns $x = Q_D(1) + A_D$, $y = Q_D(2) + A_D$ and $Q_B(1) - Q_B(2)$. The first equation is the difference between the market clearing condition (6) in state $(N,1)$ and its counterpart in state $(B,1)$:

$$\begin{aligned} & (p[1-p_N-p_B]+R)\,x + (1-p)[1-p_N-p_B]\,y - RA_D \\ = \; & \lambda p_D p x^2 + \lambda p_D (1-p)\, y^2 \\ & +\lambda p(1-p)\left[\begin{array}{c} \left[(1-p_B)^2 - (p_N)^2\right](x-y)^2 \\ +2[1-p_N-p_B](x-y)[Q_B(1)-Q_B(2)] \end{array}\right] \end{aligned} \tag{45}$$

The second equation is the difference between the market clearing condition in state $(N,2)$ and its counterpart in state $(B,2)$:

$$\begin{aligned} & (1-p)[1-p_N-p_B]\,x + (p[1-p_N-p_B]+R)\,y - RA_D \\ = \; & \lambda p_D (1-p)\, x^2 + \lambda p_D p y^2 \\ & +\lambda p(1-p)\left[\begin{array}{c} \left[(1-p_B)^2 - (p_N)^2\right](x-y)^2 \\ +2[1-p_N-p_B](x-y)[Q_B(1)-Q_B(2)] \end{array}\right] \end{aligned} \tag{46}$$

The third equation is the difference between the market clearing condition in state $(B,1)$ and its counterpart in state $(B,2)$:

$$\begin{aligned} & (2p-1)(1-p_B)(x-y) - [R-(2p-1)][Q_B(1)-Q_B(2)] \\ = \; & \lambda(2p-1)\,p_B(1-p_B)(x+y)(x-y) \end{aligned} \tag{47}$$

We start by taking the difference between (45) and (46):

$$\begin{aligned} & \left(R+(2p-1)\left[1-p_N-p_B\right]\right)(x-y) \\ = \; & \lambda\,(2p-1)\,p_D\,(x+y)\,(x-y) \end{aligned} \tag{48}$$

One solution of (48) is:

$$x=y=\frac{\kappa\pm\left[\kappa^2-4R\lambda p_D A_D\right]^{0.5}}{2\lambda p_D} \tag{49}$$

and $Q_B(1)-Q_B(2)=0$. This corresponds to the first two equilibria in Proposition 7. In each equilibria the asset price is the same in state 1 as in state 2, and only depends on whether we are in N or B. The two equilibria correspond to the fundamental and sunspot-like equilibria of Proposition 6.

When $x\neq y$ (48) implies that $\omega=x+y=Q_D\,(1)+Q_D\,(2)+2A_D$, and (47) implies that $Q_B(1)-Q_B(2)=\delta(Q_D\,(2)-Q_D\,(1))$. The sum of (45) and (46) implies that:

$$Q_D\,(2)-Q_D\,(1)=\pm\left[\eta\left(\kappa\omega-2RA_D-0.5\lambda p_D\omega^2\right)\right]^{0.5} \tag{50}$$

Equilibrium 3 is the value of (50) with the positive sign on the right-hand side and equilibrium 4 is the value with the negative sign.

If $p\rightarrow 1$, we have $\omega=\kappa\,(\lambda p_D)^{-1}$ and $Q_D\,(2)-Q_D\,(1)=\pm\left[\kappa^2-4\lambda p_D R A_D\right]^{0.5}(\lambda p_D)^{-1}$. This implies that:

$$Q_D\,(i)+A_D=\frac{\kappa\pm\left[\kappa^2-4R\lambda p_D A_D\right]^{0.5}}{2\lambda p_D}$$

In equilibrium 3 state 1 is associated with the negative value in the numerator, while it is associated with the positive value in state 2.

We now ensure that asset prices are positive in all equilibria. We first derive the conditions under which the bracket in (50) is positive. Notice that $\delta>1$ when $p=0.5$. As δ is an increasing function if p, δ is thus always above 1 This implies that $1-p_B+p_N-2\delta<-p_B-(1-p_N)<0$, and η is positive as p_N and p_B are both above 0.5. The numerator of the bracket in (50) is positive when:

$$A_D<\left[\kappa\omega-0.5\lambda p_D\omega^2\right](2R)^{-1}$$

This restriction requires that $p>\bar{p}$.

We next show that $Q_B(2)$ is the lowest value of the asset price in equilibrium 3. This equilibrium corresponds to the value of (50) with the positive sign on the right-hand side, so we have $Q_D(2) > Q_D(1)$ and $Q_B(1) - Q_B(2) = \delta(Q_D\,(2) - Q_D\,(1)) > 0$. We can show that $Q_D(1) > 0$. It then follows that $Q_D(2) > 0$, so that $Q_N(2) > Q_B(2)$. Also, it follows that $Q_N(1) > Q_B(1) > Q_B(2)$.

The final step is to show that $Q_B(2)$ is positive. The asset market clearing condition in state $(N, 1)$ can be written as:

$$Q_B(2) = \frac{A_B + \nu}{R - 1}$$

where:

$$\begin{aligned} \nu \;=\; & RA_D + (p - R)\left(Q_D\,(1) - Q_D\,(2)\right) \\ & + 0.5\omega\left(p_N - R\right) + 0.5\left(\left(2p - 1\right)p_N - R\right)\left(Q_D\,(1) - Q_D\,(2)\right) \\ & - \lambda p_N(1 - p_N)\begin{pmatrix} (0.5\omega)^2 + \left(0.5\left(Q_D\,(1) - Q_D\,(2)\right)\right)^2 \\ +0.5\left(2p - 1\right)\omega\left(Q_D\,(1) - Q_D\,(2)\right) \end{pmatrix} \\ & - \lambda p(1 - p)(p_N\left(Q_D\,(1) - Q_D\,(2)\right) + \left(Q_B\,(1) - Q_B\,(2)\right))^2 \end{aligned} \tag{51}$$

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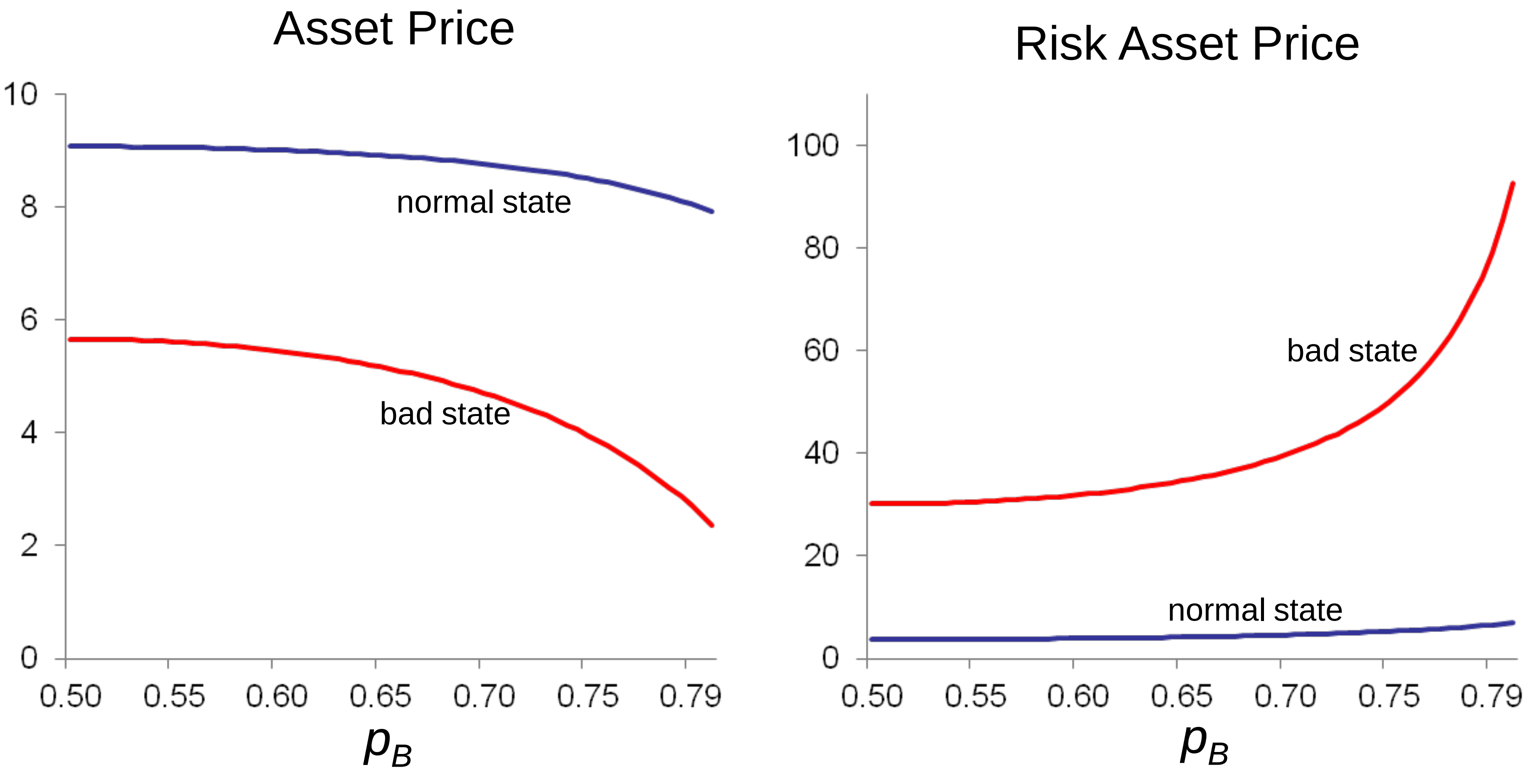
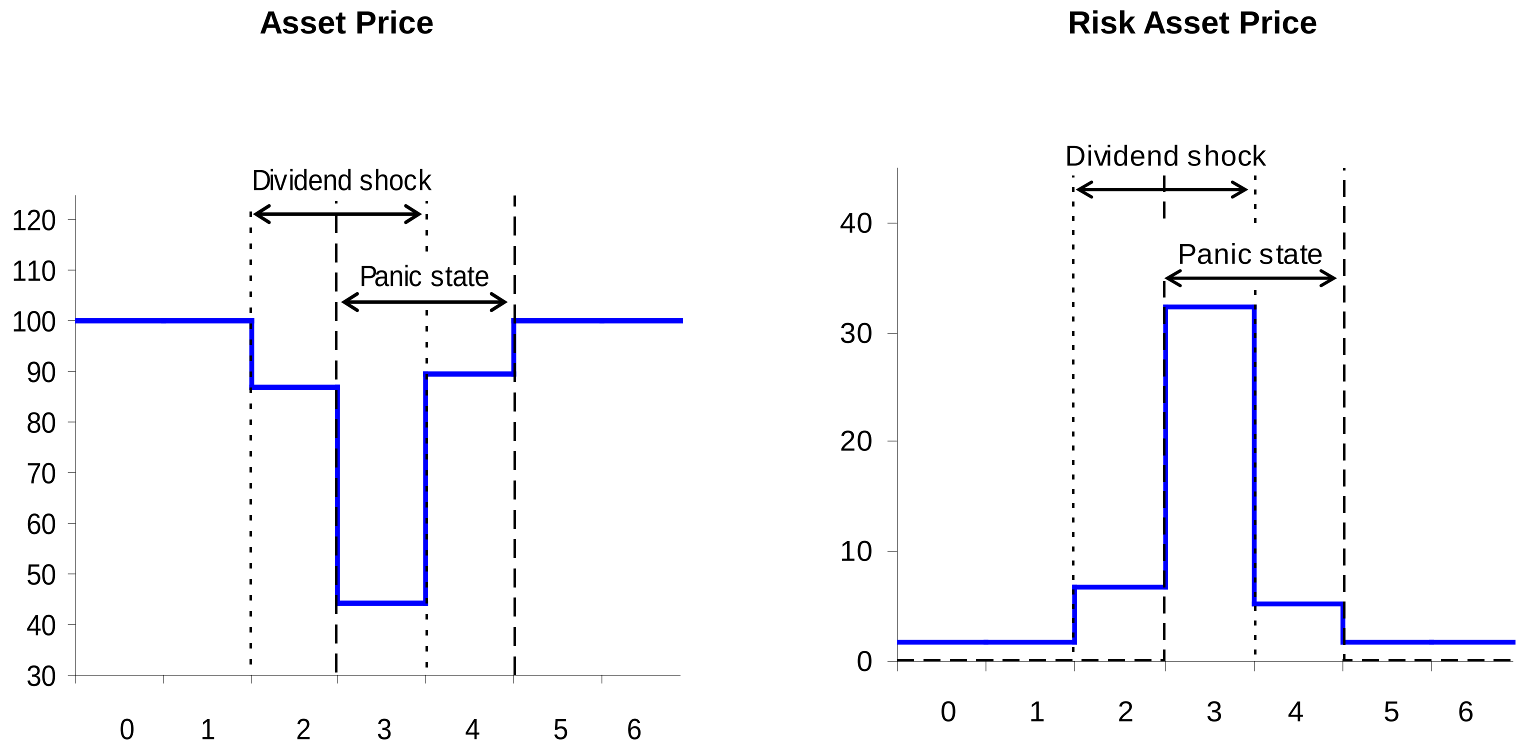
Figure 1: Illustration of Risk Panic in Markov Sunspot Equilibrium*



* The chart on the left shows the asset price in the normal state N and the bad state B as a function of the probability p_B of remaining in the bad state. The probability of remaining in the good state is held at 0.99. The chart on the right show risk, measured as the standard deviation of the asset price next period, divided by the asset price today, in both the normal and bad states. Other than p_B, which varies along the horizontal axis, the parameterization is as follows: $\overline{A} = 1; R = 1.1; p_N = 0.99; \lambda = \gamma K / W = 0.5$

Figure 2: Illustration of Risk Panic in Switching Equilibrium*

vertical slashed lines = state 2 (panic state); vertical dotted lines = dividend shock



* In periods 0 and 1 the economy is in state (N,1), where the dividend is high and there is no panic (state 1). At time 2 the dividend drops by 10% from 1 to 0.9. At time 3 there is a shift to state 2 (panic state). At time 4 the dividend rises back to 1, but we remain in the panic state. Starting with date 5 we return to state (N,1) where the dividend is high and there is no panic. The parameterization is as follows:

$A_N = 1; A_B = 0.9; R = 1.1; p_N = 0.99; p_B = 0.9; p = 0.99; \lambda = \gamma K / W = 0.5$