

Stochastic Optimization

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- We add shocks to the growth model.
- Recursive methods are needed.
- The resulting model is used to study
 - business cycles
 - asset pricing

Model

The household (or planner) maximizes

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (1)$$

subject to

$$k_{t+1} = f(k_t, \theta_t) - c_t \quad (2)$$

Productivity shocks take on N discrete values

$$\theta_t \in \{\theta^1, \dots, \theta^N\} \quad (3)$$

and follow a Markov process

$$\Pr(\theta_{t+1} = \theta^j \mid \theta_t = \theta^i) = \Omega_{ij}$$

E_0 denotes the expectation given information at date 0.

What are the choice variables?

- The planner cannot choose sequences $\{c_t, k_t\}$ because we don't know the realizations of θ_t .
- He must choose *state contingent plans*.
- For every history

$$s^t = (s_0, \dots, s_t) \quad (4)$$

where $s_t = (k_t, \theta_t)$, choose

$$c_t = c(s^t) \quad (5)$$

and

$$k_{t+1} = \kappa(s^t) \quad (6)$$

Two period example

Two period example

The problem is

$$\max E_0 \sum_{t=1}^2 \beta^{t-1} u(c_t)$$

subject to

$$k_2 = f(k_1) - c_1$$

$$c_2 = f(k_2, \theta_2)$$

$$\theta_1 \text{ known}$$

Two period example

Write out the expectation explicitly:

$$\begin{aligned} \max & \beta u(c_1) + \sum_{j=1}^N \Pr(\theta_2 = \theta_j) \beta u(c_2(\theta_j)) \\ & + \lambda_1 [f(k_1, \theta_1) - c_1 - k_2] \\ & + \sum_{j=1}^N \Pr(\theta_2 = \theta_j) \lambda_2(\theta_j) [f(k_2, \theta_j) - c_2(\theta_j)] \end{aligned}$$

The household chooses c_1, k_2 and $c_2(\theta_j)$.

- Note the constraint terms:

$$\Pr(\theta_2 = \theta_j) \lambda_2(\theta_j) [f(k_2, \theta_j) - c_2(\theta_j)] \quad (7)$$

- This looks like the household only needs to satisfy the constraint with some probability.
- Not true - the constraint must hold in every history.
- But this is just notation.
- Define $\tilde{\lambda}_2(\theta_j) = \Pr(\theta_2 = \theta_j) \lambda_2(\theta_j)$ to see this.

$$c_1 : u'(c_1) = \lambda_1$$

$$k_2 : \lambda_1 = \sum \Pr(\theta_2 = \theta_j) \lambda_2(\theta_j) f_k(k_2, \theta_j)$$

$$c_2(\theta_j) : \beta u'(c_2(\theta_j)) = \lambda_2(\theta_j)$$

Euler equation:

$$\begin{aligned} u'(c_1) &= \beta \sum \Pr(\theta_2 = \theta_j) u'(c_2(\theta_j)) f_k(k_2, \theta_j) \\ &= \beta E \{ u'(c_2) f_k(k_2, \theta_2) | \theta_1 \} \end{aligned}$$

Many periods

Many periods

A history of length t is s^t .

The household chooses $c(s^t)$ and $k(s^t)$ to maximize

$$\sum_{s^t} p(s^t) \beta^t u(c(s^t))$$

subject to

$$\begin{aligned}x(s^t) + c(s^t) &= f(k(s^t), \theta(s^t)), \quad \forall s^t \\k(s_{t+1}, s^t) &= x(s^t), \quad \forall s^t, s_{t+1}\end{aligned}$$

The last constraint ensures that $k(s^{t+1})$ is the same for all s_{t+1} .

$$\begin{aligned} & \sum_{s^t} p(s^t) \beta^t u(c(s^t)) \\ & + \sum_{s^t} \lambda(s^t) [f(k(s^t), \theta(s^t)) - x(s^t) - c(s^t)] \\ & + \sum_{s^t} \sum_{s_{t+1}} \varphi(s_{t+1}, s^t) [k(s_{t+1}, s^t) - x(s^t)] \end{aligned}$$

Note: \sum_{s^t} means: sum over all histories for all lengths t .

First-order conditions

$$c(s^t) : \beta^t p(s^t) u'(s^t) = \lambda(s^t)$$

$$x(s^t) : \lambda(s^t) = \sum_{s_{t+1}} \varphi(s_{t+1}, s^t)$$

$$k(s^t) : \lambda(s^t) f_k(s^t) = \varphi(s^t)$$

$$\begin{aligned}\varphi(s^t) &= f_k(s^t) \beta^t p(s^t) u'(s^t) \\ &= f_k(s^t) \sum_{s^{t+1}} \varphi(s_{t+1}, s^t) \\ &= f_k(s^t) \sum_{s^{t+1}} f_k(s^{t+1}) \beta^{t+1} p(s^{t+1}) u'(s^{t+1})\end{aligned}$$

This yields the usual Euler equation:

$$\begin{aligned}u'(s^t) &= \sum_{s^{t+1}} \beta f_k(s_{t+1}, s^t) \Pr(s_{t+1}|s^t) u'(s_{t+1}, s^t) \\ &= \beta E \{f_k(t+1) u'(c(t+1)) \mid s^t\}\end{aligned}$$

The point

- With uncertainty, the sequence approach is a mess.
- Two solutions:
 - 1 Recursive methods.
 - 2 A shortcut: Maximize as if one could choose sequences.

A shortcut

Let's proceed mechanically as if we were choosing sequences $\{c_t, k_t\}$:

$$\Gamma = E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ + E_0 \sum_{t=0}^{\infty} \lambda_t [f(k_t, \theta_t) - c_t - k_{t+1}]$$

This is not quite right: the budget constraint should bind state by state, not just in expectation.

- See the 2 period example.

A shortcut

- First-order conditions

$$\begin{aligned}E_0 \beta^t u'(c_t) &= E_0 \lambda_t \\E_0 \lambda_{t-1} &= E_0 \lambda_t f_k(k_t, \theta_t)\end{aligned}$$

- Euler equation:

$$E_0 u'(c_t) = \beta E_0 u'(c_{t+1}) f_k(k_{t+1}, \theta_{t+1})$$

- When date t arrives:

$$u'(c_t) = \beta E_t u'(c_{t+1}) f_k(k_{t+1}, \theta_{t+1})$$

- The point: Treating the E as a constant in the optimization problem actually yields the right result!

A shortcut

- Why does the shortcut work, even though it is entirely wrong?
- One reason: E is linear: $E(x) = \sum p(x_i) x_i$.
- The recursive approach makes this clearer...

Recursive Approach

Recursive Approach

- We generalize the DP approach introduced for deterministic problems.
- Nothing of substance changes, except there is an E in front of each equation.
- Why does nothing change?
 - Because E is a linear operator - just the sum of $\Pr(s'|s) \times \text{outcome}(s')$.
- We start by assuming that stochastic DP works as expected.
- Then we state the conditions under which it works.

Recursive Approach to the Growth Model

- State vector: $s_t = (k_t, \theta_t)$.
 - It takes some work to show that optimal policies do not depend on the entire history s^t .
- Bellman equation:

$$\begin{aligned} V(k, \theta) &= \max u(c) + \beta E V(f(k, \theta) - c, \theta') \\ &= \max u(c) + \beta \sum_{\theta'} \Pr(\theta' | \theta) V(f(k, \theta) - c, \theta') \end{aligned}$$

- First-order conditions:

$$u'(c) = \beta E V_k(k', \theta')$$

- Envelope condition:

$$V_k(k, \theta) = \beta E V_k(k', \theta') f_k(k, \theta)$$

- Euler equation:

$$\begin{aligned}u'(c) &= \beta E \{f_k(k', \theta') u'(c')\} \\u'(c[k, \theta]) &= \beta \sum_{\theta'} \Pr(\theta' | \theta) f_k(k', \theta') u'(c[k', \theta'])\end{aligned}$$

- Solution: $V(k, \theta), c(k, \theta)$ that satisfy:

- 1 Given $V, c(k, \theta)$ maximizes the right-hand-side of the Bellman equation.
- 2 V is a fixed point of the Bellman operator:
$$V(k, \theta) = u(c[k, \theta]) + \beta E V(f[k, \theta] - c[k, \theta], \theta').$$

Continuous state Markov chains

- What if the random vector takes on a continuum of values?
- Simply replace sums with integrals when calculating expectations.

Continuous state Markov chains

- Assume that $\theta \in \Theta = [\underline{\theta}, \bar{\theta}]$.
- The evolution of θ is governed by a **transition function**:

$$\Pr(\theta' \leq x | \theta) = \Pi(x, \theta) \quad (8)$$

- This is really a cdf conditional on θ .
- The transition density for this CDF is π with

$$\int_{\underline{\theta}}^x \pi(\theta', \theta) d\theta' = \Pi(x, \theta) \quad (9)$$

- This is the analogue to the transition matrix $\Pr(\theta' | \theta)$ in the discrete case.

- The conditional expectation of f is then

$$\begin{aligned} E [f (\theta') | \theta] &= \int_{\underline{\theta}}^x f (\theta') \pi (\theta', \theta) d\theta' \\ &= \int_{\underline{\theta}}^x f (\theta') \Pi (d\theta', \theta) \end{aligned}$$

- The point: In the continuous case, simply replace all the $\sum_{\theta'} \Pr (\theta' | \theta)$ with $\int \pi (\theta', \theta) d\theta'$.

Theorems: Stochastic DP

Theorems: Stochastic DP

- The assumptions needed and the results are very similar to the deterministic case.
- Acemoglu has a simplified version with discrete random variables.
- Stokey, Lucas with Prescott have more general results.

The generic problem

Environment

- Start with capital stock $x(0)$.
- The shocks $z(t)$ follow a discrete Markov chain.
 - A strong assumption.
 - It can be relaxed without affecting results too much.
 - But at the expense of notation.

The generic problem

- After each node $z^t = (z(0), \dots, z(t))$ of shocks, choose next period's capital stock

$$x(t+1) = x[z^t] \quad (10)$$

- $x(t+1)$ is constrained to lie in the set $G(x(t), z(t))$.
- Key: period utility and constraints only depend on current realizations $(x(t), z(t))$, not on history.
 - If not: DP fails (or we need tricks).

P1: Sequence problem

$$V^*(x(0), z(0)) = \max_{\{x[z^t]\}} E_0 \sum_{t=0}^{\infty} \beta^t U(x[z^{t-1}], x[z^t], z(t))$$

subject to

$$x[z^t] \in G(x[z^{t-1}], z(t))$$

$x(0)$ given

The law of motion for x is built into U .

P2: Recursive problem

$$V(x, z) = \max_{y \in G(x, z)} U(x, y, z) + \beta E[V(y, z') | z]$$

.
So much simpler!

A1:

- $G(x, z)$ is nonempty.
- For all feasible plans:

$$\lim_{n \rightarrow \infty} E \left[\sum_{t=0}^{\infty} \beta^t U(x[z^{t-1}], x[z^t], z(t)) \mid z(0) \right] < \infty.$$

A **feasible plan** is now a collection of state contingent plans ($x[z^t]$ for all histories z^t) that satisfies $x[z^t] \in G$.

- X is a compact subset of \mathbb{R}^K
 - where $x(t) \in X$.
- U is continuous.

[There are additional issues when z does not live in a finite set]

Theorem 1: Equivalence of values

- Assume A1 and A2.
- $V^*(x, z)$ in the sequence problem solves the recursive problem.
- $V(x, z)$ in the recursive problem equals $V^*(x, z)$ in the sequence problem, IF

$$\lim_{t \rightarrow \infty} \beta^t E \left[V \left(x \left[z^{t-1} \right], z(t) \right) \right] = 0$$

for all feasible plans.

Theorem 2: Principle of Optimality

- Assume A1 and A2.
- Any optimal plan in the sequence problem satisfies the Bellman equation with value V^* :

$$V^* \left(x \left[z^{t-1} \right], z(t) \right) = U \left(x \left[z^{t-1} \right], x \left[z^t \right], z(t) \right) + \beta E \left[V^* \left(x \left[z^t \right], z(t+1) \right) \right]$$

- Any feasible plan that solves the above attains V^* in the sequence problem.

The point: Solving P1 and P2 are equivalent.

Theorem 3: Existence of solutions

- Assume A1 and A2.
- An optimal plan exists for any initial conditions $x(0), z(0)$.
- V is unique, continuous, bounded in x for each z .

- $U(x, y, z)$ is strictly concave in the sense that

$$U(\bar{x}, \bar{y}, z) \geq \alpha U(x, y, z) + (1 - \alpha) U(x', y', z) \quad (11)$$

with strict inequality when $x \neq x'$, where $\bar{x} = \alpha x + (1 - \alpha) x'$ and $\bar{y} = \alpha y + (1 - \alpha) y'$.

- The set $G(x, z)$ is convex in the sense that

$$y \in G(x), y' \in G(x') \implies \bar{y} \in G(\bar{x}) \quad (12)$$

- $U(x, y, z)$ is strictly increasing in all elements of x .
- G is monotone: $x \leq x' \implies G(x, z) \subset G(x', z)$ for all z .

- U is continuously differentiable in x .

Theorem 4: Concavity of V

- Assume A1, A2, A3.
- Then V is strictly concave in x .
- The optimal plan $x[z^t] = \pi(x(t), z(t))$ is unique and π is continuous in x .

Recall A3: U strictly concave. G convex.

Then the Bellman equation is a concave optimization problem.

Theorem 5: Monotonicity of V

- Assume A1, A2, A4.
- Then V is strictly increasing in x .

Recall A4: U and G are monotone in x .

Theorem 6: Differentiability of V

- Assume A1-A3, A5.
- Then V is continuously differentiable in x .

Recall A5: U is differentiable.

Theorem 8: Euler equations

- Assume A1-A5.
- Then the interior feasible plan that satisfies the Euler equation

$$D_y U(x, \pi(x, z), z) + \beta E [D_x U(\pi(x, z), \pi(\pi(x, z), z'), z') | z] = 0$$

and the TVC

$$\lim_{t \rightarrow \infty} \beta^t E [D_x U(t) \pi(t)] = 0$$

solves the recursive problem P2.

- The Euler equation with scalar x :

$$\frac{\partial U(x, x', z)}{\partial x'} + \beta E \left[\frac{\partial U(x', x'', z')}{\partial x'} | z \right] = 0 \quad (13)$$

- Note: the TVC must hold starting from any node $x[z^t], z(t)$.

- If the shock z lives in a continuum, nothing of substance changes.
- Acemoglu 16.4

Example: Permanent income hypothesis

Example: Permanent income hypothesis

- A household maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c(t)) \quad (14)$$

- u has all nice properties: strictly increasing, concave, differentiable.
- Budget constraint:

$$a(t+1) = (1+r)a(t) + w(t) - c(t) \quad (15)$$

- $w(t)$ is i.i.d. with $\Pr(w(t) = w_j) = q_j$.

Budget constraint

- A tricky issue: the budget constraint.
- How much should the household be able to borrow?
- A natural borrowing constraint:

$$a(t) \geq - \sum_{s=0}^{\infty} \frac{w_1}{(1+r)^s} \equiv -b_1 \quad (16)$$

- This ensures that the household can repay his debts, even if he receives the worst possible income realization in each period.
- Because w is i.i.d. and the household lives forever, b_1 is a constant.

Sequence problem

- The history is w^t .
- The household chooses $c[w^t]$ for all possible histories.
- The problem is too tedious to even write down...

$$V(a, w) = \max_{a' \in [-b_1, (1+r)a+w]} u([1+r]a + w - a') + \beta EV(a', w') \quad (17)$$

Mapping into the generic problem:

- $x \rightarrow a, z \rightarrow w$
- $G(z, x) \rightarrow [-b_1, (1+r)a + w]$
- $U(x, y, z) \rightarrow u([1+r]a + w - a')$

First-order conditions

Verify A1-A5 ...

Then we can characterize the solution by the FOCs:

$$u'(c) = \beta EV_a(a', w') \quad (18)$$

$$V_a(a, w) = (1 + r) u'(c) \quad (19)$$

Euler:

$$u'(c) = \beta (1 + r) Eu'(c') \quad (20)$$

- $G(x, z) = [-b_1, (1+r)a + w]$ is nonempty – b_1 is constructed that way.
- For all feasible plans:
$$\lim_{n \rightarrow \infty} E \left[\sum_{t=0}^{\infty} \beta^t U(x[z^{t-1}], x[z^t], z(t)) \mid z(0) \right] < \infty.$$
 - This is NOT generally satisfied.
 - $(1+r) > \beta$ could imply unbounded growth.
 - We need a restriction that r not too high. Tedious details...

- X is a compact subset of \mathbb{R}^K
 - Here: $X = \mathbb{R}_+$ which is obviously not compact.
 - We need to argue that bounding a from above does not bind (when $1 + r < \beta$).
- U is continuous – by assumption.

- $U(x, y, z)$ is strictly concave in the sense that

$$U(\bar{x}, \bar{y}, z) \geq \alpha U(x, y, z) + (1 - \alpha) U(x', y', z) \quad (21)$$

with strict inequality when $x \neq x'$, where $\bar{x} = \alpha x + (1 - \alpha) x'$ and $\bar{y} = \alpha y + (1 - \alpha) y'$.

- Here: $U([1 + r][\alpha a_1 + (1 - \alpha) a_2] + w - [\alpha a'_1 + (1 - \alpha) a'_2])$ with $\partial U / \partial a' < 0$ and $\partial^2 U / \partial (a')^2 < 0$.
- The set $G(x, z)$ is convex in the sense that

$$y \in G(x), y' \in G(x') \implies \bar{y} \in G(\bar{x}) \quad (22)$$

- easy to check

- $U(x, y, z)$ is strictly increasing in all elements of x .
 - Here: $\partial u / \partial a > 0$.
- G is monotone: $x \leq x' \implies G(x, z) \subset G(x', z)$ for all z .
 - Here: $(1 + r)a$ is increasing in a .

- U is continuously differentiable in x . – by assumption.

Quadratic case

- Assume $u(c) = \phi c - 0.5c^2$
- $u'(c) = \phi - c$.
- Euler:

$$\phi - c = \beta(1+r)E[\phi - c'] \quad (23)$$

- Nothing in the info set at t should predict consumption growth [in this example: $E c' - c$]. Hall 1978.
- Strangely, a large literature has tested this prediction, even though it only holds with quadratic utility!

- We could have derived the Euler equation naively by treating E as a constant in the optimization problem.
- The deterministic FOCs turn out to be correct (in many [all?] cases).

- Acemoglu, "Introduction to Modern Economic Growth," ch. 16.1-16.2.
- Krusell, "Lecture notes for Macroeconomics I, 2004," ch. 6.
- Stokey & Lucas with Prescott (1989) discuss the technical details of stochastic Dynamic Programming.
- Sargent & Ljungqvist, ch. 2 talk about Markov chains.