

# Stochastic Growth Model

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November 17, 2010

# Stochastic Growth Model

- We add shocks to the growth model.
- Recursive methods are needed.
- The resulting model is used to study
  - business cycles
  - asset pricing

- The history is of shocks is  $\theta^t$ .
- Preferences:

$$\sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \Pr(\theta^t | \theta_0) u(c[\theta^t]) \quad (1)$$

- Technology: for all  $\theta'$

$$X[\theta^t] = F(K[\theta^t], L) + (1 - \delta)K[\theta^t] - c[\theta^t] \quad (2)$$

$$K[\theta^t, \theta'] = X[\theta^t] \quad \forall \theta' \quad (3)$$

# Bellman equation

Define  $k = K/L$ .

$$V(k, \theta) = \max_{k' \in [0, f(k, \theta) + (1 - \delta)k]} u(f(k, \theta) + (1 - \delta)k - k') \quad (4)$$

$$+ \beta E [V(k', \theta') | \theta] \quad (5)$$

# First-order conditions

- Verify that A1-A5 hold ... Theorems 1-6 apply.

- FOC

$$u'(c) = \beta E V_k(k', \theta')$$

- Envelope

$$V_k(k, \theta) = u'(c) [f_k(k, \theta) + 1 - \delta]$$

- Euler

$$u'(c) = \beta E [u'(c') \{f_k(k', \theta') + 1 - \delta\} | \theta] \quad (6)$$

- Solution:  $V(k, \theta)$  and  $\pi(k, \theta)$  that "solve" the Bellman equation

- Now for the bad news ... there really isn't much one can say about the solution analytically.
- But see Campbell (1994) for a discussion of a log-linear approximation.

The model comes in 2 flavors.

## ① Complete markets

- for every history, there exists an asset that pays in that state of the world
- the implication is complete risk sharing: all idiosyncratic risks are insured
- aggregate risks remain

## ② Incomplete markets

- some securities are missing
- there is no representative agent

# Trading arrangements

- With complete markets, date 1 Arrow-Debreu trading is convenient
  - Uncertainty essentially disappears from the model.
- With incomplete markets, it is easiest to specify the set of securities available at each date.
  - Sequential trading.

# Complete markets - Arrow Debreu trading

- The environment is standard.
- The history is of shocks is  $\theta^t$ .
- **Arrow securities** pay out 1 unit of the good exactly when state  $\theta^t$  occurs.
- The point: This looks like a static model without uncertainty.

$$\sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \Pr(\theta^t | \theta_0) u(c[\theta^t]) \quad (7)$$

- Expenditures in state  $\theta^t$ :

$$x(\theta^t) = p(\theta^t) [c(\theta^t) + s(\theta^t)] \quad (8)$$

- $p(\theta^t)$  is the price of the good in state  $\theta^t$ .
- $c$  is consumption
- $s$  is "saving:" buy goods (capital) and rent to firms.

- Income in state  $\theta^t$ :

$$y(\theta^t) = w(\theta^t) + R(\theta^t) s(\theta^{t-1}) \quad (9)$$

- $w(\theta^t)$  is the wage.
- $R(\theta^t)$  is the payoff from renting a unit of the good to the firm.
- Both are state contingent.

# Household: budget constraint

- Lifetime budget constraint:

$$\sum_{t=0}^{\infty} \sum_{\theta^t} [y(\theta^t) - x(\theta^t)] + p(\theta_0) s_0 = 0 \quad (10)$$

- $s_0$  is the initial endowment of goods.
- With complete markets, there is a lifetime budget constraint, even under uncertainty.
  - Because there really is no uncertainty any more.
  - At each node, the household's spending and income are fully predictable.

- Firms maximize the total value of profits.
  - There is no discounting because of Arrow-Debreu trading.
- Profits in state  $\theta^t$ :

$$p(\theta^t) [F(K[\theta^t], L[\theta^t], \theta_t) + (1 - \delta)K[\theta^t]] - R(\theta^t)K(\theta^t) - w(\theta^t)L(\theta^t)$$

- Value of the firm: sum of profits over all states.
- FOCs are standard: since the firm does not own anything, it maximizes profits state-by-state.

# Competitive Equilibrium

- Allocation:  $c(\theta^t), s(\theta^t), K(\theta^t), L(\theta^t)$ .
- Price system:  $p(\theta^t), w(\theta^t), R(\theta^t)$  for all histories  $\theta^t$ .
- These satisfy:
  - 1 Household optimality.
  - 2 Firm optimality.
  - 3 Market clearing:
    - $L(\theta^t) = 1$ .
    - $K(\theta^t, \theta_{t+1}) = s(\theta^t)$ .
    - Goods market.

# Competitive Equilibrium

## Comments

- This looks like a static model without uncertainty.
  - Each history defines new goods: output, labor, capital rental.
- The setup is far more complicated than the recursive one.
- We could add (Arrow) securities. They would be redundant.

- We set up the C.E. with sequential trading.
- We need **Arrow securities**.
- Each security,  $a(\theta^{t+1})$  is indexed by the state of the world in which it pays off:  $\theta^{t+1}$ .
- The asset is purchased for price  $\bar{p}(\theta^t, \theta')$  in state  $\theta^t$ .
- It pays one unit of consumption if  $\theta^{t+1} = [\theta^t, \theta']$ .

- Budget constraint:

$$c(\theta^t) + s(\theta^t) = w(\theta^t) + a(\theta^t) + R(\theta^t)k(\theta^t) \quad (11)$$

$$s(\theta^t) = \sum_{\theta_{t+1}} \bar{p}(\theta^t, \theta_{t+1}) a(\theta^t, \theta_{t+1}) + x(\theta^t) \quad (12)$$

$$k(\theta^t, \theta_{t+1}) = x(\theta^t) \quad (13)$$

- Numeraire: consumption at each node  $\theta^t$ .

- Household problem:

$$\max \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \Pr(\theta^t | \theta_0) u(c[\theta^t]) \quad (14)$$

s.t. budget constraints for all  $\theta^t$ .

# Recursive household problem

- State:  $(\vec{a}, k, \theta)$ .
  - $\vec{a}$ : holdings of all the  $a(\theta)$ .
- Given prices:  $w$  and  $\bar{p}(\theta, \theta')$ .
- Bellman equation:

$$V(\vec{a}, k, \theta) = \max_{c, a'(\theta'), k'} u(c) + \beta \sum_{\theta'} q(\theta'|\theta) V(\vec{a}', k', \theta')$$

s.t. budget constraint

$$\sum_{\theta'} \bar{p}(\theta, \theta') a'(\theta') + k' + c = w + a(\theta) + Rk$$

- For  $a'(\theta')$ :

$$u'(c) \bar{p}(\theta, \theta') = \beta q(\theta'|\theta) \frac{\partial V(\bar{a}'[\theta'], k', \theta')}{\partial a(\theta')} \quad (15)$$

- For  $k'$ :

$$u'(c) = \beta \sum_{\theta'} q(\theta'|\theta) \frac{\partial V(\bar{a}', k', \theta')}{\partial k'} \quad (16)$$

# First order conditions

- Envelope:

$$\partial V(\vec{a}, k, \theta) / \partial a(\theta) = u'(c) \quad (17)$$

$$\partial V(\vec{a}, k, \hat{\theta}) / \partial a(\theta) = 0 \quad (18)$$

$$\partial V(\vec{a}, k, \theta) / \partial k = u'(c) R \quad (19)$$

- Euler equation holds state by state for state contingent claims:

$$u'(c) \bar{p}(\theta, \theta') = \beta q(\theta' | \theta) u'(c[a'(\theta'), \theta']) \quad (20)$$

- Euler equation for capital:

$$\begin{aligned} u'(c) &= \beta \sum_{\theta'} q(\theta' | \theta) R(\theta, \theta') u'(c[a'(\theta'), k', \theta']) \\ &= \beta E R' u'(c') \end{aligned} \quad (21)$$

- Since capital can be replicated by buying a set of Arrow securities:

$$\sum_{\theta'} \bar{p}(\theta, \theta') R(\theta, \theta') = 1 \quad (22)$$

- Proof: Solve (20) for  $q(\theta'|\theta)$  and substitute into (21).

- We can write down a sequential equilibrium definition, similar to the Arrow-Debreu.
  - Everything is indexed by  $\theta^t$ .
- More powerful: Recursive Competitive Equilibrium.
  - Everything is a function of the current state.

- Define an aggregate state vector:  $S = (\theta, K)$ .
  - In general: we need to keep track of the distribution of  $(\theta_i, k_i)$  across households.
  - Here: all households are identical.
- The law of motion for the aggregate state:

$$\begin{aligned}\Pr(\theta'|\theta) &= q(\theta'|\theta) \\ K' &= G(\theta, K)\end{aligned}$$

where  $G$  is endogenous.

- Given:
  - aggregate state and its law of motion.
  - price functions:  $w(S), R(S)$  and  $\bar{p}(S, \theta')$ .
- Bellman equation:

$$V(\vec{a}, k, S) = \max_{c, a'(\theta'), k'} u(c) + \beta \sum_{\theta'} q(\theta' | \theta) V(\vec{a}'[\theta'], k', S')$$

s.t. budget constraint

$$\sum_{\theta'} \bar{p}(\theta, \theta') a'(\theta') + k' + c = w(S) + a(\theta) + R(S)k$$

and aggregate law of motion

$$S' = G(S)$$

- First-order conditions: unchanged.
- Solution:  $V(a, k, S)$  and policy functions  $c(a, k, S)$ ,  $k' = \kappa(a, k, S)$ .

- Always the same because the firm has a static problem:
- Solution:  $R(S), w(S)$ .

- Equilibrium objects:

- ① Household: Value function and policy functions.
- ② Firm: Price functions.
- ③ Aggregate law of motion:  $K' = G(\theta, K)$ .

- Equilibrium conditions:

- ① Household optimality.
- ② Firm optimality.
- ③ Market clearing.
- ④ Consistency:

$$G(\theta, K) = \kappa(K, \theta, K) \tag{23}$$

where the household's policy function is  $k' = \kappa(k, \theta, K)$ .

- Note: We could toss out all the Arrow securities without changing anything.
- The model boils down to:
  - 1 Euler equation for  $K$ :  $u'(c) = \beta E [R' u'(c')]$
  - 2 Law of motion for  $K$ :  $K' = F(K, L) + (1 - \delta)K - c$ .
  - 3 FOC:  $R = F_K(K, L) + 1 - \delta$ .
- This changes when individuals are not identical.

# Recursive CE

What do we gain?

- Avoid having to carry around infinite histories.
- Equilibrium contains few objects.
  - Especially when the economy is **stationary**.
- All endogenous objects are functions.
  - Results from functional analysis can be used to determine their properties.
- Recursive CE is easy to compute.

- What if agents are heterogeneous?
- With complete markets, risk is perfectly shared.
- The simplest case: An endowment economy with Arrow-Debreu trading.
- The state is  $\theta^t$ .

- There are  $I$  types of households, indexed by  $i$ .
- Endowments are  $y^i(\theta^t)$ .
- Preferences are

$$\sum_t \sum_{\theta^t} \beta^t q(\theta^t) u^i(c^i[\theta^t])$$

- Budget constraints:

$$\sum_t \sum_{\theta^t} p(\theta^t) [c^i(\theta^t) - y^i(\theta^t)] = 0 \quad (24)$$

- First-order conditions are as usual:

$$q(\theta^t) \beta^t \frac{\partial u^i(c^i[\theta^t])}{\partial c^i[\theta^t]} = \lambda_i p(\theta^t) \quad (25)$$

where  $\lambda_i$  is the Lagrange multiplier.

- Complete risk sharing: For all  $\theta^t$  the MRS is equated across households:

$$MRS(\theta^t, \hat{\theta}^\tau) = -\frac{\beta^t \partial u^i(c^i[\theta^t]) / \partial c^i[\theta^t]}{\beta^\tau \partial u^i(c^i[\hat{\theta}^\tau]) / \partial c^i[\hat{\theta}^\tau]} = \frac{p(\theta^t) / q(\theta^t)}{p(\hat{\theta}^\tau) / q(\hat{\theta}^\tau)} \quad (26)$$

- Equivalently, the ratio of marginal utilities between 2 agents is the same for all  $\theta^t$ :

$$\frac{\partial u^i(c^i[\theta^t]) / \partial c^i[\theta^t]}{\partial u^j(c^j[\theta^t]) / \partial c^j[\theta^t]} = \frac{\lambda_i}{\lambda_j} \quad (27)$$

- If households have identical preferences and there is no aggregate uncertainty (the aggregate endowment is the same in all states), then individual consumption is constant.

- Acemoglu, "Introduction to Modern Economic Growth," ch. 16-17.
- Stokey & Lucas with Prescott (1989) discuss the technical details of stochastic Dynamic Programming.
- Ljungqvist & Sargent, ch. 2 talk about Markov chains. Ch. 7 covers complete market economies (Arrow-Debreu and sequential trading). Ch. 6: Recursive CE.
- Campbell, John Y. (1994). "Inspecting the mechanism: an analytical approach to the stochastic growth model." *Journal of Monetary Economics* 33: 463-506.