

# Review Questions: Stochastic Growth Model

Prof. Lutz Hendricks . November 17, 2010

Useful problems from the texts:

- Acemoglu, "Introduction to Modern Economic Growth:" Exercises 17.2.1, 17.5.2, 17.9, 17.11, 17.12, 17.18, 17.19.
- Romer, "Advanced Macro:" Exercises 4.8, 4.9, 4.10, 4.11.

## 1 Stochastic growth model

[Due to Oksana Leukhina] Consider a neoclassical growth model with uncertainty. Time is discrete:  $t = 0, 1, 2, \dots$

Demographics: There is a unit mass of identical households.

Preferences:  $\mathbb{E} \sum_{t=0}^{\infty} \beta^t \ln c_t$ .

Endowments: The household is endowed with one unit of labor time in each period and with initial capital stock  $k_0$ .

Technology: Output per capita is given by  $\theta_t k_t^\alpha$  where  $k$  is the capital-labor ratio. The resource constraint is given by  $k_{t+1} = \theta_t k_t^\alpha - c_t$ .  $\theta_t$  is an i.i.d. random variable that takes on the values  $\theta^1$  with probability  $\pi$  and  $\theta^2$  with probability  $1 - \pi$ .

- a. State the social planner's problem in recursive form.
- b. Solve for the value function and the policy function. Start from the guess  $V(k, \theta^i) = A + B \ln k + C \ln \theta^i$ .
- c. Define a Competitive Equilibrium, assuming: (i)  $\theta_t$  is an aggregate shock; (ii) trading takes place in a sequence of markets; (iii) households trade a complete set of Arrow securities (an Arrow security pays one unit of the good if history  $\theta^t = (\theta_1, \dots, \theta_t)$  occurs).

### 1.1 Answer: Stochastic growth model

[I have not checked the details.]

- a. The Bellman equation is formulated as

$$V(k, \theta_i) = \max_{c, k' \geq 0} \{ \ln(c) + \beta \mathbb{E} V_1(\theta', k') \}$$
$$\text{s.t. } c + k' \leq \theta_i k^\alpha,$$

where  $\theta$  is a discrete i.i.d. r.v. that takes on two values  $\theta_1$  with probability  $\pi$  and  $\theta_2$  with probability  $(1 - \pi)$ .

Find the value function using the method of underdetermined coefficients. Use  $V(k, \theta_i) = A + B \ln k + C \ln \theta_i$  as a guess.

The Bellman equation is formulated as

$$V(k, \theta_i) = \max_{k'} \{ \ln(\theta_i k^\alpha - k') + \beta [\pi V(k', \theta_h) + (1 - \pi) V(k', \theta_l)] \}.$$

Plugging the guess into the functional equation gives

$$\begin{aligned} V(k, \theta_i) &= \max_{k'} \{ \ln(\theta_i k^\alpha - k') + \beta [\pi (A + B \ln k' + C \ln \theta_h) + (1 - \pi) (A + B \ln k' + C \ln \theta_l)] \} = \\ &= \max_{k'} \{ \ln(\theta_i k^\alpha - k') + \beta B \ln k' + \beta A + \beta C [\pi \ln \theta_h + (1 - \pi) \ln \theta_l] \}. \end{aligned}$$

Taking first-order conditions, which are necessary and sufficient, gives

$$\frac{-1}{(\theta_i k^\alpha - k')} + \frac{\beta B}{k'} = 0.$$

The optimal policy function is then

$$k' = \frac{\beta B}{1 + \beta B} \theta_i k^\alpha,$$

i.e., a constant share of output is saved in each period.

We have

$$A + B \ln k + C \ln \theta_i = \ln \left( \frac{\theta_i k^\alpha}{1 + \beta B} \right) + \beta B \ln \left( \frac{\beta B}{1 + \beta B} \theta_i k^\alpha \right) + \beta A + \beta C [\pi \ln \theta_h + (1 - \pi) \ln \theta_l]$$

$$A + B \ln k + C \ln \theta_i = \ln \left( \frac{\theta_i k^\alpha}{1 + \beta B} \right) + \beta B \ln \left( \frac{\beta B}{1 + \beta B} \theta_i k^\alpha \right) + \beta A + \beta C [\pi \ln \theta_h + (1 - \pi) \ln \theta_l]$$

$$A + B \ln k + C \ln \theta_i = \alpha (1 + \beta B) \ln k + (1 + \beta B) \ln \theta_i - (1 + \beta B) \ln (1 + \beta B) + \beta B \ln \beta B + \beta A + \beta C [\pi \ln \theta_h + (1 - \pi) \ln \theta_l]$$

We are now able to determine the coefficients:

$$B = \alpha (1 + \beta B), \text{ i.e. } B = \frac{\alpha}{(1 - \alpha \beta)}$$

$$C = (1 + \beta B)$$

$$A = ((1 + \beta B) \ln \theta_i - (1 + \beta B) \ln (1 + \beta B) + \beta B \ln \beta B + \beta C [\pi \ln \theta_h + (1 - \pi) \ln \theta_l]) / (1 - \beta)$$