

# Bewley Models

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August 10, 2009

- For many applications we need models with **heterogeneous agents**.
- In Bewley models, agents are ex ante identical.
- They are ex post heterogeneous because they are hit by idiosyncratic shocks.
- Incomplete markets prevents sharing these risks.

# An Endowment Economy

- Demographics:
  - There is a unit measure of households.
  - Each lives forever.

- Preferences:

$$E \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (1)$$

- Technology:
  - Households receive random endowments  $y_t \in Y$  (finite).
  - Transition matrix:  $\pi(y'|y)$ .

# No aggregate uncertainty

- Assume a "law of large numbers."
- Let  $\Pi(y)$  be the stationary distribution of  $y$ .
- Assume that the fraction of households with endowment  $y$  is  $\Pi(y)$ .
- The aggregate endowment  $\bar{y}$  is constant over time.
- With complete markets, households would not face any uncertainty.

# Household problem

- Flow budget constraint:

$$a' = y + (1 + r)a - c \quad (2)$$

- Borrowing constraint:

$$a' \geq -b \quad (3)$$

# Household problem

- Focus on a **stationary equilibrium**.
  - Meaning: Aggregates & prices are constant over time.
- State vector:  $(a, y)$ .
- Given:  $r$ .
- Bellman equation:

$$v(a, y) = \max u(c) + \beta \sum_{y'} \pi(y'|y) v(a', y') \quad (4)$$

s.t. budget constraint and borrowing constraint.

# Household problem I

- The borrowing constraint may bind. We need Kuhn-Tucker.
- Bellman equation

$$v(a, y) = \max u(y + (1+r)a - a') \quad (5)$$

$$+ \beta \sum_{y'} \pi(y'|y) v(a', y') + \mu(a' + b) \quad (6)$$

- First-order conditions:

$$u'(c) = \beta \sum \pi(y'|y) \frac{\partial v(a', y')}{\partial a'} + \mu \quad (7)$$

$$\partial v / \partial a = u'(c) (1+r) \quad (8)$$

$$\mu(a' + b) = 0 \quad (9)$$

- Euler equation

$$u'(c) \geq E\beta(1+r)u'(c') \quad (10)$$

with equality if  $a' > -b$ .

- Solution:  $v(a, y), c(a, y), a'(a, y)$  that satisfy the usual conditions.

# Household problem

How does the household behave?

- For a given  $y$ , if "cash on hand"

$$x = y + (1 + r) a \quad (11)$$

is sufficiently high: Choose  $a' > -b$  and satisfy standard Euler equation.

- If "cash on hand" is below a cutoff, set  $a' = -b$  and "violate" the Euler equation.
  - The borrowing constraint depresses current consumption.

# Stationary Recursive Competitive Equilibrium

- Aggregate state: The joint distribution of assets and endowments:  $\Phi(a, y)$ .
  - This is needed to compute aggregates.
- Objects:
  - Household:  $v(a, y), c(a, y), a'(a, y)$ .
  - $\Phi(a, y)$ .
  - Price function:  $r(\Phi)$ .
- Equilibrium conditions:
  - Household: see above.
  - Market clearing.
  - Time invariance of  $\Phi$ .

# Stationary Recursive Competitive Equilibrium

Market clearing

- Goods:

$$\int \int c(a, y) \Phi(da, dy) = \int y \Pi(dy) \quad (12)$$

- Bonds:

$$\int \int a'(a, y) \Phi(da, dy) = 0 \quad (13)$$

# Stationary Recursive Competitive Equilibrium

Time invariance of the distribution

- Informally, household choices determine tomorrow's distribution  $\Phi'$ .
- The policy function  $a'(a, y)$  implies a law of motion for  $\Phi$ .
- In stationary equilibrium, the law of motion must imply  $\Phi' = \Phi$ .

# Law of motion for the distribution

- Define a transition function  $Q((a, y), (A, Y))$ .
- Its value is the probability (mass) of households in state  $(a, y)$  today that end up in  $(a', y') \in (A, Y)$  tomorrow.

$$Q((a, y), (A, Y)) = \begin{cases} \sum_{y' \in Y} \pi(y'|y) & \text{if } a'(a, y) \in A \\ 0 & \text{otherwise} \end{cases}$$

- This is because  $a'$  is deterministic.

- Law of motion

$$\Phi'(A, Y) = \int \int Q((a, y), (A, Y)) \Phi(da, dy) \quad (14)$$

- In words:

- $\Phi'(A, Y)$ : What is the mass of households in the set of states  $(A, Y)$  tomorrow?
- For any  $(a, y)$ , this is given by  $Q((a, y), (A, Y))$ .
- Sum over all  $(a, y)$  to get the total mass.
- Stationarity then means:  $\Phi'(A, Y) = \Phi(A, Y)$  for all  $(A, Y)$ .

# Non-stationary Equilibrium

- The equilibrium concept generalizes easily to economies where  $\Phi$  evolves over time.
- Household:
  - Add the aggregate state  $\Phi$  to the household's state:  $v(a, y, \Phi)$  and  $a'(a, y, \Phi)$ .
  - The household takes prices as functions of the aggregate state:  $r(\Phi)$ .
  - The household knows the law of motion for  $\Phi$ :  $\Phi' = H(\Phi)$ .
- Equilibrium:
  - Drop stationarity of  $\Phi$ .

# Where is this useful?

- Models of the wealth distribution:
  - Krusell, Per; Anthony A. Smith (1998). "Income and wealth heterogeneity in the macroeconomy." *Journal of Political Economy* 5: 867-96.
- Models of business cycles with heterogeneous agents:
  - Rios-Rull, Jose-Victor (1996). "Life-Cycle Economies and Aggregate Fluctuations." *Review of Economic Studies* 63(3): 465-89.

- Acemoglu, "Introduction to Modern Economic Growth," ch. 17.4.
- Krueger, "Macroeconomic Theory," ch. 10.