

# Review Questions: Asset Pricing

Econ720. Fall 2022. Prof. Lutz Hendricks

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## 1 Many Trees

Consider the standard Lucas fruit tree model. Assume that household  $i$  initially holds  $K_i$  trees of type  $i$ . Each tree produces a dividend  $d_{it}$  that is independently drawn from some distribution  $G(d)$ .

1. Define a competitive equilibrium.
2. Characterize the optimal portfolio held by each household.
3. Show that every household enjoys constant consumption, even without having access to Arrow securities.

Note: The answer is kind of obvious. I have not formally derived it. So I don't know how hard that is.

## 2 Lucas Trees

Consider a version of Lucas's asset pricing model with one representative household who is endowed with one tree.

**Technology:** The tree produces a stream of dividends,  $d_t$ , where  $d_0 = 1$ . The dividend growth rate,  $\frac{d_{t+1}}{d_t}$ , can take on one of two values,  $\mu + \sigma$  or  $\mu - \sigma$ , where  $\mu > 1$ . The dividend growth rate is a Markov chain with transition matrix  $P$ . In particular, assume  $P$  is a symmetric matrix where the probability of *switching* growth rates is  $p$ , where  $p \in (0, 1)$ .

**Preferences:** Household preferences are given by  $E \sum_{t=0}^{\infty} \beta^t \ln(c_t)$ .

**Markets:** At each date, there are markets for consumption goods, trees, and state-contingent claims that pay one unit of consumption tomorrow in a particular state of the world.

**Questions:** (a) Define a solution to the household problem. Think carefully about what the household's state variables are.

(b) Define a recursive competitive equilibrium.

(c) Solve for the equilibrium pricing function for trees. You should find that the price-dividend ratio for trees is constant over time.

(d) Solve for the pricing functions for state-contingent claims. (Assume  $\lim_{n \rightarrow \infty} \beta^{n+1} E_t x_{t+n} = 0$  to rule out bubbles).

(e) Add a riskless bond to this economy (a sure claim to one unit of the consumption good next period). Compute the price of a riskless bond. Hint: There is no need to resolve for the equilibrium price functions (why not?).

(f) Now assume  $p = 0.5$ . Compute the *average* rate of return on bonds and trees. What is the equity premium for this economy?

## 2.1 Answer: Lucas Trees

(a) Let  $\mu(z) = \frac{d}{d-1}$ . The household's state variables are the exogenous states  $(z, d)$  and last period's choices  $s, y(0), y(1)$ . The controls are  $s', y'(0), y'(1)$ . The Bellman equation is

$$v(s, y(0), y(1), z, d) = \max_{s', y'(1), y'(0)} \{ \ln c + \beta E [v(s', y'(0), y'(1), z', d')] \}$$

subject to the budget constraint

$$\begin{aligned} c + p(z, d) s' + y'(1) q(1|z) + y'(0) q(0|z) &= s(p(z, d) + d) + y(z) \\ d' &= \mu(z') d \end{aligned}$$

More explicitly

$$\begin{aligned} v(s, y(0), y(1), z, d) &= \max u \left( s(p(z, d) + d) + y(z) - p(z, d) s' - \sum_{z'} y'(z') q(z'|z) \right) \\ &\quad + \beta \sum_{z'} \Pr(z'|z) v(s', y'(0), y'(1), z', d\mu(z')) \end{aligned}$$

FOCs:

$$p(z, d) u'(c) = \beta \sum_{z'} \Pr(z'|z) v_1(s', y'(0), y'(1), z', \mu(z') d) \quad (1)$$

$$q(z'|z) u'(c) = \beta \Pr(z'|z) v_{y'(z')}(s', y'(0), y'(1), z', \mu(z') d) \quad (2)$$

where the latter holds for all  $z'$ . Envelope conditions:

$$v_1(\cdot) = u'(c) (p(z, d) + d) \quad (3)$$

$$v_{y(z)}(\cdot) = u'(c) \quad (4)$$

The Euler equation is standard:

$$u'(c) p(z, d) = \beta \sum_{z'} \Pr(z'|z) u'(c') [p(z', d\mu(z')) + d\mu(z')]$$

There is another Euler equation for state-contingent claims:

$$u'(c) q(z'|z) = \beta \Pr(z'|z) u'(c')$$

where it is understood that  $c'$  is the realization in the right state tomorrow.

A solution to the household problem consists of a value function and policy functions for  $s'(s, y(0), y(1), z, d)$  and  $y'(z'; s, y(0), y(1), z, d)$  such that:

- $v$  satisfies the fixed-point property of the Bellman equation, given optimal policies.
- Optimal policies maximize the right hand side of the Bellman equation, given  $v$ .

(b) A recursive competitive equilibrium is:

1. A set of individual decision rules,
2. A set of pricing functions  $p(z, d), q(z'|z)$ ,

such that:

1. Given pricing functions the decision rules solve the household's dynamic program (see (a)).
2. Markets clear:

$$\begin{aligned} s'(1, 0, 0, z, d) &= 1 \\ y'(z'; z, 0, 0, z, d) &= 0 \end{aligned}$$

(c) It's convenient to use time subscripts now:

$$\frac{p_t}{c_t} = \beta E_t \left[ \frac{1}{c_{t+1}} (p_{t+1} + d_{t+1}) \right]$$

In equilibrium  $c_t = d_t$ . Hence,

$$\frac{p_t}{d_t} = \beta E_t \left[ \frac{1}{d_{t+1}} (p_{t+1} + d_{t+1}) \right]$$

Define  $x = p/d$ . Iterating forward and using law of iterated expectations then yields

$$\begin{aligned} x_t &= \beta + \beta^2 + \beta^2 E_t [E_{t+1} \{x_{t+1}\}] \\ &= \beta + \beta^2 + \beta^3 + \dots + \beta^{n+1} + \beta^{n+1} E_t E_{t+1} \dots E_{n-1} [E_n \{x_n\}] \\ &= \beta + \beta^2 + \beta^3 + \dots + \beta^{n+1} + \beta^{n+1} E_t x_{t+n} \end{aligned}$$

Assuming  $\lim_{n \rightarrow \infty} \beta^{n+1} E_t x_{t+n} = 0$  yields

$$\begin{aligned} \frac{p_t}{d_t} &= \frac{\beta}{1 - \beta}, \text{ or} \\ p(z, d) &= \frac{\beta}{1 - \beta} d \end{aligned} \tag{5}$$

The price-dividend ratio is constant, verifying our assumption that  $x$  depends only on  $z$ .

(d) Next, (2) and (4) imply

$$q(z'|z) = \beta \Pr(z'|z) \frac{d}{\mu(z') d}$$

because, in equilibrium  $c = d \implies \frac{c}{c} = \frac{d}{d}$ . Hence,

$$\begin{aligned} q(z' = 1|z = 0) &= \frac{\beta p}{\mu + \sigma}; \\ q(z' = 1|z = 1) &= \frac{\beta(1-p)}{\mu + \sigma}; \end{aligned}$$

Similarly,

$$\begin{aligned} q(z' = 0|z = 0) &= \frac{\beta p}{\mu - \sigma}; \\ q(z' = 0|z = 1) &= \frac{\beta(1-p)}{\mu - \sigma}; \end{aligned}$$

(e) The key is that the bond is a redundant asset. We can therefore determine the price,  $Q(z)$ , simply as the sum of two assets that replicate the bond:

$$Q(z) = q(z' = 1|z) + q(z' = 0|z), \tag{6}$$

which implies that

$$\begin{aligned} Q(z = 0) &= \frac{\beta p}{\mu + \sigma} + \frac{\beta(1-p)}{\mu - \sigma}; \\ Q(z = 1) &= \frac{\beta(1-p)}{\mu + \sigma} + \frac{\beta p}{\mu - \sigma} \end{aligned}$$

(f)  $p = 0.5$ . Then  $Q(z = 0) = Q(z = 1) = \frac{\beta\mu}{\mu^2 - \sigma^2} = Q$  (say). Denote the average gross and net rate of returns on bond, as  $R^b$  and  $r^b$ , respectively. Then using (6)

$$\begin{aligned} R^b &= \frac{1}{Q} \\ &= \frac{\mu^2 - \sigma^2}{\beta\mu}; \\ r^b &= R^b - 1 \end{aligned}$$

Denote the average gross and net rate of returns on trees, as  $R^s$  and  $r^s$ , respectively. Then

$$\begin{aligned} R^s &= E_t \left( \frac{p_{t+1} + d_{t+1}}{p_t} \right) \\ &= E_t \left( \frac{\frac{\beta}{1-\beta} d_{t+1} + d_{t+1}}{\frac{\beta}{1-\beta} d_t} \right) \\ &= \frac{1}{\beta} E_t \left( \frac{d_{t+1}}{d_t} \right) \\ &= \frac{\mu}{\beta} \\ r^s &= R^s - 1 \end{aligned}$$

The equity premium

$$\begin{aligned} \gamma &= R^s - R^b \\ &= \frac{\mu}{\beta} - \frac{\mu^2 - \sigma^2}{\beta\mu} \\ &= \frac{\sigma^2}{\beta\mu} \end{aligned}$$