

1 Two stocks

Based on a problem due to Steve Williamson. Consider an endowment economy inhabited by a representative consumer with preferences

$$E \sum_{t=0}^{\infty} \beta^t u(c_t) \quad (1)$$

where u is strictly increasing, strictly concave, and satisfies $u'(0) = \infty$. The household owns shares in two firms, indexed by $j = 1, 2$. In each period, i.i.d. random variables $\theta_{j,t}$ are drawn. Firm j then "produces" (really, receives and endowment of) $\theta_{j,t} y$ and pays out this amount as a dividend. $y > 0$ is a constant.

In each period, households trade shares of the two firms at price $p_{j,t}$. Shares are traded after the $\theta_{j,t}$ are realized and dividends are paid. Households also trade a riskless bond in zero net supply with endogenous price q_t .

1. State the household problem in recursive form.
2. Derive the Euler equations. Assume an interior solution where the household holds both stocks.
3. From now on assume that $\sum_j \theta_j = 1$. Assume that there are no bubbles. Solve for the equilibrium asset prices.
4. Derive the equity premium. Explain your finding.
5. Now assume that y_t is random and utility is $u(c) = \ln(c)$. Solve for the equilibrium stock prices and expected returns. What can you say about the connection between expected stock returns and growth, i.e., are stocks expected to do well when there are strong growth prospects? What is the intuition for this?

1.1 Answer: Two stocks

1. Household problem. Let $\Theta = (\theta_1, \theta_2)$. The Bellman equation is given by

$$V(s_1, s_2, b; \Theta) = \max u(c) + \beta E[V(s'_1, s'_2, b'; \Theta') | \Theta] \quad (2)$$

subject to the budget constraint

$$\sum_j s_j [\theta_j y + p_j] + b = \sum_j s'_j p_j + qb' + c \quad (3)$$

2. Euler equations. First order conditions yield standard Lucas asset pricing equations:

$$u'(c) = \beta E \left[u'(c') \frac{\theta'_j y' + p'_j}{p_j} \right] \quad (4)$$

$$u'(c) = \beta E [u'(c') / q] \quad (5)$$

3. Equilibrium prices. c is constant at y . Therefore,

$$p_j = \beta E [\theta'_j y + p'_j | \Theta] \quad (6)$$

$$q = \beta \quad (7)$$

Since θ is i.i.d., stock prices are constant over time:

$$p_j (1 - \beta) = \beta y E [\theta_j] \quad (8)$$

4. Equity premium. Expected gross stock returns equal β^{-1} . The equity premium is zero. The two stocks are jointly riskless.

5. Random endowment. Use the standard asset pricing equation:

$$p_{j,t} = E \sum_{k=1}^{\infty} \beta^k \frac{u'(c_{t+k})}{u'(c_t)} [\theta_{j,t+k} y_{t+k}] \quad (9)$$

Using $c_t = y_t$, this simplifies to

$$p_{j,t} = y_t E \sum_{k=1}^{\infty} \beta^k \theta_{j,t+k} \quad (10)$$

$$= y_t E (\theta_j) \frac{\beta}{1 - \beta} \quad (11)$$

with expected return $\frac{1}{\beta} E \left[\frac{y'}{y} \right]$. Expected returns are high in states with rising expected y , i.e. when $E(y'/y)$ is high. Intuition: When y' is expected to be high relative to today's y , the same is true for consumption. Marginal utility behaves in the opposite way. A high expected return is needed to entice the household to save in such a state.