Asset pricing

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Issues

1. What determines the rates of return / prices of various assets?
2. How can risk be measured and priced?

- We use the Lucas fruit tree model.
- The implications are far more general than the simple model.

The Lucas Fruit Tree Model

- We study the model introduced by Lucas (1978).
- **Agents**:
  - A single representative household.
- **Preferences**:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

- E_0 is the expectation as of time $t = 0$.

- This is an endowment economy.
- There are K identical fruit trees.
- Each tree yields d_t units of consumption goods in period t.
- d_t is random and the same for all trees.
- Trees cannot be produced.
- Fruits cannot be stored.

- The aggregate resource constraint:

$$c_t = Kd_t \tag{2}$$

- Assume that d is a finite Markov chain with transition matrix $\pi(d', d)$.
- An important feature: All uncertainty is **aggregate**.
- There are no opportunities for households to insure each other.
- This is why we can work with a representative household.

- There are markets for fruits and for trees.
- There is also a one period bond, issued by households (in zero net supply).
    - Its purpose is to determine a risk-free interest rate.

Household problem

- The household starts out with bonds (b_0) and shares (k_0).
- At each date, he chooses c_t, b_{t+1}, k_{t+1}.
- The **budget constraint** is

$$p_t \, k_{t+1} + b_{t+1} = R_t \, b_t + (p_t + d_t) \, k_t - c_t \tag{3}$$

- Notation:
  - p : the price of trees. Suppressing dependence on the state.
  - R : the real interest rate on bonds.
  - the price of bonds is normalized to 1 (how?).

Household problem

$$V(k, b, d) = \max u(c) + \beta EV(k', b', d') \quad (4)$$

subject to

$$Rb + (p + d)k - c + pk' - b' = 0 \quad (5)$$

Household problem

First-order conditions:

$$\begin{aligned} c &: u'(c) = \lambda \\ k' &: \lambda p = EV_k(k', b', d') \\ b' &: \lambda = EV_b(k', b', d') \end{aligned}$$

Envelope:

$$\begin{aligned} V_k &= \lambda(p + d) \\ V_b &= \lambda R \end{aligned}$$

Household problem

Euler equations

$$u'(c_t) = \beta E_t \left\{ u'(c_{t+1}) R_{t+1} \right\} = \beta E_t \left\{ u'(c_{t+1}) \frac{p_{t+1} + d_{t+1}}{p_t} \right\}$$

Household problem

Solution

- A solution consists of state contingent plans $\{c(d^t), k(d^t), b(d^t)\}$ for all histories d^t.
- These satisfy:
  - 2 Euler equations
  - 1 budget constraint.
  - b_0 and k_0 given.
  - Transversality: $\lim_{t\to\infty} E_0 \beta^t u'(c_t) \; [b_t + p_t k_t] = 0$.

Market clearing

For every history we need:
Bonds:

$$b_t = 0$$

Trees:

$$k_t = K_t$$

Goods:

$$c_t = K_t d_t$$

There is no trade in equilibrium!

Competitive Equilibrium

- A CE consists of:
  1. an allocation: $\{c(d^t), b(d^t), k(d^t)\}$.
  2. a price system: $\{p(d^t), R(d^t)\}$
- These satisfy:
  1. household: 2 Euler equations and 1 budget constraint.
  2. 3 market clearing conditions.

Recursive Competitive Equilibrium

Objects:

- Solution to the household problem: $V(k, b, d)$ and $c(k, b, d)$, $k' = \kappa(k, b, d)$, $b' = B(k, b, d)$.
- Price functions: $p(d), R(d)$.

Equilibrium conditions:

- Household: 4
- Market clearing: 2
- No need for consistency: law of motion of the aggregate state is exogenous.

Consumption smoothing

- The Euler equation implies (for any asset):

$$E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} = 1 \tag{6}$$

- Define: Marginal rate of substitution:

$$MRS_{t+1} = \beta \, u'(c_{t+1}) / u'(c_t) \tag{7}$$

- MRS_{t+1} is inversely related to consumption growth.
- With $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$:

$$u'(c) = c^{-\sigma} \tag{8}$$

$$MRS_{t+1} = \beta \, (c_{t+1}/c_t)^{-\sigma} \tag{9}$$

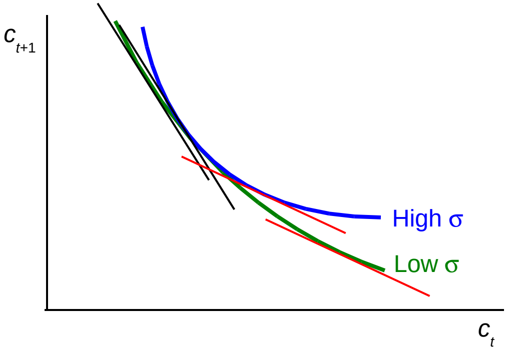
- With constant R, the household chooses constant consumption growth.

Consumption smoothing

- The coefficient of relative risk aversion (σ) determines how much *MRS* fluctuates with c.
- High σ implies that the household chooses smooth consumption.
- Illustration for the deterministic case:

Consumption smoothing

- With high σ, marginal utility changes a lot when c changes.
- The household then keeps c smooth.



Asset pricing implications

- We will now derive the famous **Lucas asset pricing equation**.
- Define: Rate of return on trees: $R^S_{t+1} = (p_{t+1} + d_{t+1}) / p_t$.
- Directly from the 2 Euler equations:

$$E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R_{t+1} \right\} = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} R^S_{t+1} \right\} = 1$$

- Or

$$E \{ MRS_{t+1} \, R_{t+1} \} = E \left\{ MRS_{t+1} \; R^S_{t+1} \right\} = 1 \tag{10}$$

When does an asset pay a high expected return?

Re-write asset pricing equation using

$$Cov(x,y) = E(x\,y) - E(x)\,E(y)$$

as

$$\begin{aligned} 1 &= E\{MRS\}\;E\{R\} + Cov\,(MRS, R) \\ E(R) &= \frac{1 - Cov(MRS, R)}{E(MRS)} \end{aligned} \tag{11}$$

When do assets pay high returns?

$$E(R) = \frac{1 - Cov(MRS, R)}{E(MRS)} \quad (12)$$

- High returns require low $Cov\left(MRS, R\right)$.
- With CRRA utility: high MRS means low consumption growth.
- Therefore: Assets pay high returns if their returns are **positively correlated with consumption growth**.

When do assets pay high returns?

Intuition

- Imagine there are good times (high c) and bad times (low c).
- There are 2 assets: A pays dividends in good times, B pays in bad times.
- The value of the dividend is $u'(c)$.
- Assets that pay in good times are not valuable: $u'(c)$ is low.
- Assets that pay in bad times provide insurance - they are valuable (have low expected returns).

Risk (premia)

- The "risk free" assets has expected return

$$E\left(R_f\right) = \frac{1}{E\left(MRS\right)} \tag{13}$$

- A "risky" asset has expected return

$$E\left(R\right) = \frac{1 - Cov(MRS, R)}{E(MRS)} \tag{14}$$

- The **risk premium** is

$$E\left(R\right) - E\left(R_f\right) = -\frac{Cov(MRS, R)}{E(MRS)} \tag{15}$$

- This defines what **risk** means: **covariance with consumption growth**.
- Note that risk can be **negative** (insurance).

The Equity Premium Puzzle

- Mehra and Prescott (1985): Asset return data pose a puzzle for the theory.
- The equity premium is "high" (6-7% p.a.)
- The cov of c growth and R_s is low.
  - The reason: Consumption is very smooth.

The Equity Premium Puzzle

TABLE 1
SUMMARY STATISTICS
UNITED STATES ANNUAL DATA, 1889–1978

	Sample Means		
R_t^s	0.070		
R_t^b	0.010		
C_t/C_{t-1}	0.018		

	Sample Variance-Covariance		
	R_t^s	R_t^b	C_t/C_{t-1}
R_t^s	0.0274	0.00104	0.00219
R_t^b	0.00104	0.00308	–0.000193
C_t/C_{t-1}	0.00219	–0.000193	0.00127

Kocherlakota (1996)

The Equity Premium Puzzle

A back-of-the envelope calculation with CRRA utility:

$$EP = -\frac{Cov\left(\beta\left[c_{t+1}/c_t\right]^{-\sigma},\ R_s\right)}{E\left\{\beta\left[c_{t+1}/c_t\right]^{-\sigma}\right\}} \tag{16}$$

Take log utility: $\sigma = 1$.

- $Cov\left(MRS, R_s\right) \simeq -0.0022$.
- $E\left(MRS\right) \simeq 1$.
- $EP \simeq 0.2\%$.
- Replicating the observed equity premium requires **very high risk aversion** $(\sigma = 40)$.

Investors forego very large returns.

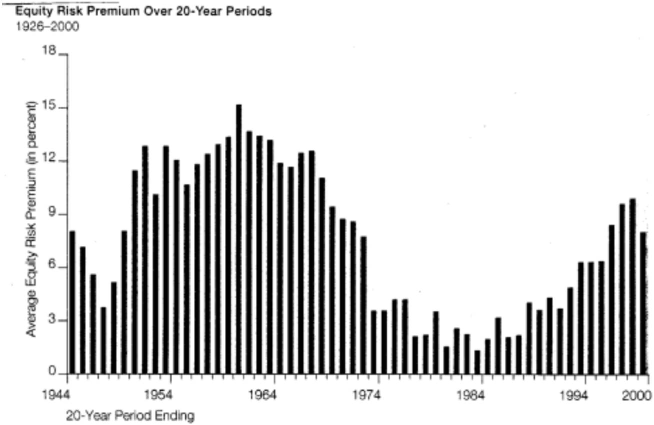
Table 3

Terminal value of $1 invested in Stocks and Bonds

Investment Period	**Stocks**		**T-bills**	
	Real	Nominal	Real	Nominal
1802-1997	$558,945	$7,470,000	$276	$3,679
1926-2000	$266.47	$2,586.52	$1.71	$16.56

Mehra & Prescott (2003)

Over 20 year holding periods: stocks dominate bonds.



Mehra & Prescott (2003)

- The EP puzzle shows that we do not understand
  1. what households view as "risky"
  2. why households place a high value on smooth consumption
- This has implications for:
  1. The welfare costs of business cycles
     - They are very low in standard models.
  2. Stock price volatility.
     - Standard models fail to explain it (see below).

Proposed explanations include:

1. Habit formation: $u\left(c_t, c_{t-1}\right) = \frac{\left[c_t - \gamma c_{t-1}\right]^{1-\sigma}}{1-\sigma}$.
   - Implies high risk aversion when c_t is close to c_{t-1}.
2. Heterogeneous agents
   - Implicit in the standard model: all idiosyncratic risk is perfectly insured.
3. Borrowing constraints
   - The young should hold stocks (long horizon), but cannot.
   - The old receive mostly capital income and find stocks risky.
4. Taxes / regulations (McGrattan & Prescott 2001)
   - The runup in stock prices since the 1960s stems from lower dividend taxes & laws permitting institutional investors to hold equity.

Beta

- Now we derive the famous "beta" measure of risk.
- Suppose asset m (the market) is perfectly correlated with marginal utility:

$$u'(c_{t+1}) = -\gamma \; R_{m,t+1} \tag{17}$$

- The market's expected return is

$$E \; R_m - R = -\frac{Cov\,(MRS, R_m)}{E\,(MRS)} \tag{18}$$

Beta

- Now we relate the covariance term to marginal utility:

$$Cov\left(MRS, R_m\right) = Cov\left(\frac{u'\left(c_{t+1}\right)}{u'\left(c_t\right)}, R_{m,t+1}\right) = \frac{Cov\left(u'\left(c_{t+1}\right), R_{m,t+1}\right)}{u'\left(c_t\right)} \quad (1$$

$$E\left(MRS\right) = \frac{E\left(u'\left(c_{t+1}\right)\right)}{u'\left(c_t\right)} \quad (2$$

- Therefore:

$$E\left(R_m\right) - R = -\frac{Cov\left(u'\left(c_{t+1}\right), R_{m,t+1}\right)}{E\ u'\left(c_{t+1}\right)} = -\frac{\gamma\ Var\left(R_{m,t+1}\right)}{E\ u'\left(c_{t+1}\right)} \quad (21)$$

Beta

- For any asset i:

$$E\ R_i - R = -\frac{Cov\left(u'\left(c_{t+1}\right), R_i\right)}{E\ u'\left(c_{t+1}\right)} = \frac{\gamma\ Cov\left(R_m, R_i\right)}{E\ u'\left(c_{t+1}\right)} \tag{22}$$

- Take the ratio for assets i and m:

$$\begin{aligned} E\ R_i - R &= \beta_i\ \left[E\ R_m - R\right] \\ \beta_i &= \frac{Cov\left(R_m, R_i\right)}{Var\left(R_m\right)} \end{aligned}$$

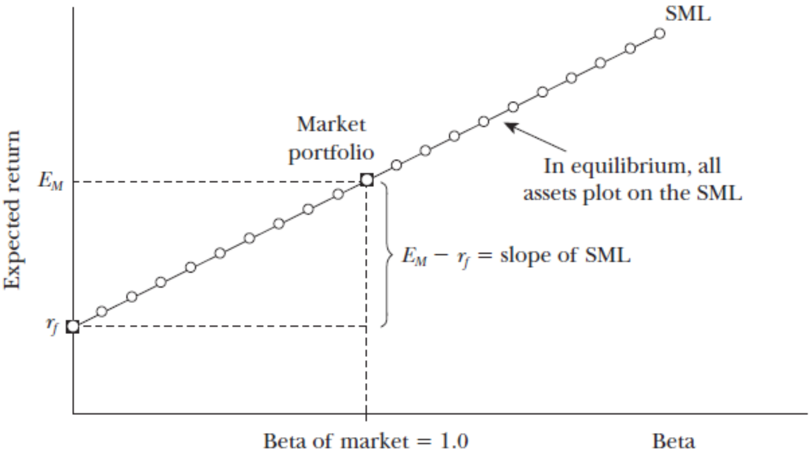
- This is the famous CAPM asset pricing equation.

Beta

- The risk premium for asset i depends on:
  - it's **beta** (essentially the correlation with the market)
  - the market price of risk: $E\ R_m - R$.
- A stock's beta can be estimated from data on past returns of the stock (R_i) and the market (using a broad stock index).
- Betas are used to
  - Measure the risk of an asset.
  - Calculate the required rate of return for investment projects.
  - Evaluation of mutual fund managers.

Securities market line

In equilibrium, all asset returns line up on the SML.

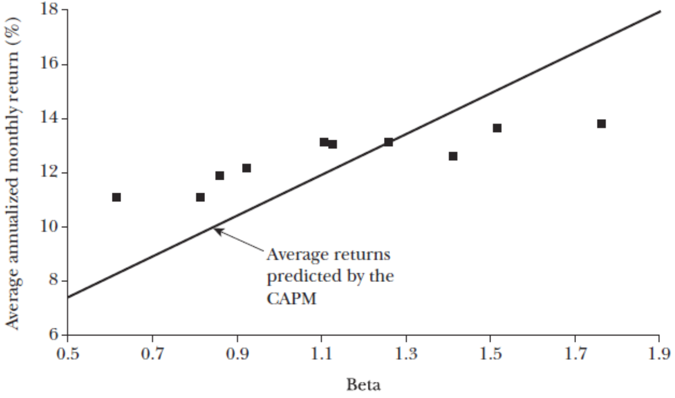


Perold (2004)

Securities market line: Evidence

Stocks with higher βs have higher expected returns, but not enough.

Average Annualized Monthly Return versus Beta for Value Weight Portfolios Formed on Prior Beta, 1928–2003



Fama (2004)

Solving for the asset price

- We show that the asset price equals the present discounted value of dividends.
- Start from the Euler equation:

$$u'(c_t) = \beta E_t \left\{ u'(c_{t+1}) \frac{p_{t+1} + d_{t+1}}{p_t} \right\} \tag{23}$$

- Solve for the price:

$$p_t = E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\} \tag{24}$$

- Replace p_{t+1} with (24) shifted to $t+1$:

$$\begin{aligned} p_t &= E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} d_{t+1} \right\} + \\ & E_t \left\{ \frac{\beta u'(c_{t+1})}{u'(c_t)} E_{t+1} \left[\frac{\beta u'(c_{t+2})}{u'(c_{t+1})} \right] (p_{t+2} + d_{t+2}) \right\} \end{aligned} \tag{25}$$

Solving for the asset price

The law of iterated expectations:

$$E_t\left\{E_{t+1}(x)\right\} = E_t(x) \tag{26}$$

Eliminate the E_{t+1}:

$$p_t = E_t\left\{\frac{\beta\, u'(c_{t+1})}{u'(c_t)}\, d_{t+1}\right\} + E_t\left\{\frac{\beta^2\, u'(c_{t+2})}{u'(c_t)}\, (p_{t+2} + d_{t+2})\right\} \tag{27}$$

Iterate forward for T periods:

$$p_t = E_t\left\{\sum_{j=1}^{T} \frac{\beta^j\, u'(c_{t+j})}{u'(c_t)}\, d_{t+j}\right\} \tag{28}$$

$$+E_t\left\{\frac{\beta^{T+1}\, u'(c_{t+T+1})}{u'(c_{t+T})}\, (p_{t+T+1} + d_{t+T+1})\right\} \tag{29}$$

Solving for the asset price

- Impose that the last term vanishes in the limit:

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\} \tag{30}$$

- There is no good reason for this assumption!
- The asset price equals the **discounted present value of dividends**.
- The stochastic **discount factor** is the marginal rate of substitution.

Example: Log Utility

- In the Lucas model, assume: $u(c) = \ln(c)$. $K = 1$.
- In equilibrium: $c_t = d_t$.
- $MRS_{t+1} = \frac{\beta \; u'(c_{t+1})}{u'(c_t)} = \frac{\beta \; d_t}{d_{t+1}}$.
- The asset pricing equation becomes

$$\begin{aligned} p_t &= E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j \, d_t}{d_{t+j}} \, d_{t+j} \right\} \\ &= d_t \, \frac{\beta}{1-\beta} \end{aligned}$$

Example: Periodic dividends

- In the Lucas model, assume:
  - Utility is $u(c) = c^{1-\sigma}/(1-\sigma)$.
  - d_t alternates between d^H and d^L.
  - Asset pricing equation:

$$\begin{aligned} p_t &= \sum \beta^j \left(d_t / d_{t+j}\right)^{\sigma} d_{t+j} \\ &= d_t^{\sigma} \sum \beta^j d_{t+j}^{1-\sigma} \end{aligned} \tag{31}$$

- On good days, p_t is pulled up by low $u'(c')$, but is pushed down by low d_{t+1}.

The Excess Volatility Puzzle

- Consider a stock with dividend process d_t.
- Its price is given by

$$p_t = E_t \left\{ \sum_{j=1}^{\infty} \frac{\beta^j u'(c_{t+j})}{u'(c_t)} d_{t+j} \right\} \tag{32}$$

- In the data:
  - Dividends are very smooth (a goal of company policy).
  - Stock prices are much more volatile than dividends.
- But in the theory: stock prices should be the **average** of future dividends and thus **smoother** than dividends.
- This is the flip-side of the Equity Premium Puzzle.

Bubbles

- Recall how the asset pricing formula is derived:
- We iterate forward on the asset pricing Euler equation

$$p_t = E_t \left\{ \frac{\beta\, u'(c_{t+1})}{u'(c_t)} (p_{t+1} + d_{t+1}) \right\} \tag{33}$$

- We assume that the p_{t+1} term vanishes in the limit.
- What if it does not vanish?
- Then **any** (current) **asset price** can satisfy the asset pricing equation.
- The deviation between p_t and the **fundamental price** from (33) is called a **bubble**.
- It is purely a self-fulfilling expectation.

Bubbles: Example

- Consider an asset that pays no dividends.
- Its **fundamental price** is 0.
- Assume that the MRS is constant at $\frac{\beta \ u'(c_{t+1})}{u'(c_1)} = 1$.
- The the asset pricing equation is

$$p_t = E_t p_{t+1} \tag{34}$$

- One price process that satisfies this: p doubles with probability 1/2 and drops to 0 otherwise.
- This satsifies (34) for **any** p_t.
- Bubbles are a possible explanation for asset price volatility.
- Note that the bubble does not offer any excess return opportunities.

State contingent claims

- Some assets pay out only in particular states of the world
  - e.g. insurance contracts
- Standard asset pricing formulas apply to those assets.
- It just adds notation...

State contingent claims

- We start from the Lucas fruit tree model.
- In addition to stocks and bonds, households can purchase assets that pay out in exactly one state of the world.
- Notation:
  - quantity purchased of asset that pays out in state d': $y'(d')$.
  - price of that asset: $q(d'|d)$.

Household

- states: all assets held, k, b, and all $y(d)$.
- choices: b', k', $y(d')$.
- Dynamic Program:

$$V(k, b, y(1), ..., y(N), d) = \max u(c) \tag{35}$$
$$+\beta EV(k', b', y'(1), ..., y'(N), d') \tag{36}$$

subject to

$$Rb + (p + d)k + y(d) = c + b' + pk' + \sum_{d'} q(d'|d)\; y'(d') \tag{37}$$

First-order conditions for state contingent claims

$$u'(c)\, q(d'|d) = \beta \Pr(d'|d)\, V_{y(d')}\left(k', b', y'(1), ..., y'(N), d'\right) \quad (38)$$

Envelope:

$$V_{y(d)}(., d) = u'(c) \quad (39)$$

$$V_{y(d)}(., \hat{d}) = 0, \quad \hat{d} \neq d \quad (40)$$

Euler equation

$$u'(c)\, q(d'|d) = \beta \Pr(d'|d)\, u'(c') \tag{41}$$

In more standard form:

$$1 = \Pr(d'|d) \frac{\beta u'(c')}{u'(c)} \frac{1}{q(d'|d)} \tag{42}$$

where the rate of return on the state contingent claim is $1/q$.

Reading

- Romer, Advanced Macroeconomics, ch. 7.5
- Ljunqvist & Sargent, ch. 7.
- Mehra, Rajnish; Edward C. Prescott (1985). "The Equity Premium: A Puzzle." *Journal of Monetary Economics* 15(2): 145-61.
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