

Overlapping Generations

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Overlapping Generations

- What do we want in a macro model?
 - Capital accumulation, growth, ... things the Solow model has
- But we also want microfoundations.
- The simplest type of model assumes that households live forever.
 - ... but we need tools for that
- Now we study models in which households **live for 2 periods**.

What we do in this section

- How to set up and solve an OLG model
- Show that the world is **not efficient**: households may save too much.
- “Social security” can prevent overaccumulation
- We can make households "infinitely lived" by adding altruistic **bequests**.

What we don't do in this section

- We sidestep some technical issues:
 - why is there a representative household?
 - why is there a representative firm?
- We come back to those later.

An OLG Model Without Firms

An OLG Model Without Firms

Demographics

- At each date a cohort of size

$$N_t = N_0(1 + n)^t$$

is born.

- Each person lives for two periods.
- Therefore, at each date there are N_t young and N_{t-1} old households.
- An important missing market: **there cannot be any intergenerational borrowing and lending.**
 - For example, the young at t cannot borrow from the old because the old won't be around at $t+1$ to have their loans repaid.
 - If households live for more periods, the problem becomes weaker, but does not go away.

An OLG Model Without Firms

- Endowments
 - Young households receive endowments w_t .
- Technology:
 - Endowments can be stored.
 - Storing s_t today yields $f(s_t)$ tomorrow.
- Markets:
 - Goods are traded in spot markets.
 - Households can issue one period bonds with interest rate r_{t+1} .

Household Problem

- The utility function is $u(c_t^y) + \beta u(c_{t+1}^o)$.
- The budget constraints are

$$\begin{aligned}w_t &= c_t^y + s_{t+1} + b_{t+1} \\c_{t+1}^o &= f(s_{t+1}) + b_{t+1}(1 + r_{t+1})\end{aligned}$$

- Lifetime budget constraint:

$$w_t - c_t^y - s_{t+1} = [c_{t+1}^o - f(s_{t+1})] / [1 + r_{t+1}]$$

Household Problem

Lagrangian

$$\Gamma = u(c_t^y) + \beta u(c_{t+1}^o) \\ + \lambda \{ [w_t - c_t^y - s_{t+1}] - [c_{t+1}^o - f(s_{t+1})] / [1 + r_{t+1}] \}$$

FOCs:

$$u'(c_t^y) = \lambda \\ \beta u'(c_{t+1}^o) = \lambda / (1 + r_{t+1}) \\ f'(s_{t+1}) = 1 + r_{t+1}$$

Household Problem

Euler equation

$$\beta f'(s_{t+1})u'(c_{t+1}^o) = u'(c_t^y)$$

- Bond holdings follow residually from the period 1 budget constraint.
- Solution: A vector $(c_t^y, c_{t+1}^o, s_{t+1}, b_{t+1})$ which satisfies
 - 2 FOCs (an EE and the foc for s)
 - 2 budget constraints.
- This is (unsurprisingly) the same as in the two-period model.

A CE is an allocation $\{c_t^y, c_t^o, s_t, b_t\}$ and a price system $\{r_t\}$ that satisfy:

- 4 household conditions
- bond market clearing: $b_t = 0$;
- goods market clearing:

$$N_t(c_t^y + s_{t+1}) + N_{t-1}c_t^o = N_t w_t + N_{t-1}f(s_t)$$

There is no trade in equilibrium.

A Production Economy

A Production Economy

- The model is modified by adding firms who rent capital and labor from households.
- The endowment w is now interpreted as labor earnings.
- Households supply one unit of labor inelastically to firms when young.
- Capital depreciates at rate δ .

- Budget constraints:

$$\begin{aligned}w_t &= c_t^y + s_{t+1} + b_{t+1} \\c_{t+1}^o &= (s_{t+1} + b_{t+1})(1 + r_{t+1})\end{aligned}$$

- There are no profits b/c the technology has constant returns to scale.
- The lifetime budget constraint is

$$w_t - c_t^y = c_{t+1}^o / [1 + r_{t+1}]$$

$$\Gamma = u(c_t^y) + \beta u(c_{t+1}^o) + \lambda \{w_t - c_t^y - c_{t+1}^o / [1 + r_{t+1}]\}$$

FOCs:

$$\begin{aligned} u'(c_t^y) &= \lambda \\ \beta u'(c_{t+1}^o) &= \lambda / (1 + r_{t+1}) \end{aligned}$$

Euler:

$$u'(c_t^y) = \beta(1 + r_{t+1})u'(c_{t+1}^o)$$

Solution: A vector $(c_t^y, c_{t+1}^o, s_{t+1}, b_{t+1})$ that satisfies 2 budget constraints and 1 EE.

We lack one equation! Why?

Firms maximize current period profits taking factor prices (r, w) as given. The technology has constant returns to scale and satisfies Inada conditions.

Objective function:

$$\max F(K, L) - wL - qK$$

FOCs:

$$q = F_K(K, L)$$

$$w = F_L(K, L)$$

It is almost always convenient to write the production function in **intensive form**,

$$\begin{aligned} F(K, L) &= LF(K/L, 1) \\ &= Lf(k^F) \end{aligned}$$

where $k^F = K/L$ and

$$f(k^F) = F(k^F, 1)$$

This, of course, requires constant returns to scale.

Firms: Intensive form

Now the factor prices are

$$F_K = Lf'(k^F)(1/L)$$

and

$$\begin{aligned} F_L &= f(k^F) + Lf'(k^F)(-K/L^2) \\ &= f(k^F) - f'(k^F)k^F \end{aligned}$$

Therefore:

$$\begin{aligned} q &= f'(k^F) \\ w &= f(k^F) - k^F f'(k^F) \end{aligned}$$

Important: q is the rental price of capital, which differs from the interest rate r .

The solution to the firm's problem is a pair (K, L) so that the 2 FOCs hold.

Market clearing

Capital rental:

$$K_t = N_{t-1} s_t$$

Labor rental:

$$L_t = N_t$$

Bonds:

$$b_t = 0$$

Goods:

$$F(K_t, L_t) = N_t(c_t^y + s_{t+1}) + N_{t-1}(c_t^o - s_t[1 - \delta])$$

Another way of writing goods market clearing, which we will see often:

$$K_{t+1} = (1 - \delta)K_t + F(K_t, L_t) - C_t$$

Competitive Equilibrium

An allocation $(c_t^y, c_t^o, s_t, b_t, K_t, L_t)$ and a price system (q_t, r_t, w_t) that satisfy:

- the household EE and budget constraints (3 equations)
- the firm's FOCs (2 equations)
- the market clearing conditions (4 equations)

We have 9 objects and 9 equations – one is missing.

We need an accounting identity linking r and q :

- The household receives $1 + r_{t+1} = q_{t+1} + 1 - \delta$ per unit of capital.
- Therefore, $r = q - \delta$.

Saving Function and Dynamics

Saving Function and Dynamics

We need to describe how the economy responds to shocks.

For this, we need a difference equation describing how the state variables evolve over time (a law of motion).

What are the state variables?

- Variables carried over into the current period from the last period.
- Variables that are predetermined in the current period.

Here: the state variable is K_t .

More conveniently, we use $k_t = K_t/N_t$ as the state variable.

The evolution of k is characterized by the capital market clearing condition $K_t = N_{t-1} s_t$ or

$$\begin{aligned} K_t/N_t &= N_{t-1}/N_t \cdot s_t \\ (1+n)k_t &= s_t \end{aligned} \tag{1}$$

together with the household saving function

$$s_{t+1} = s(w_t, r_{t+1}) \tag{2}$$

Saving function

Start from the Euler equation

$$\beta(1 + r_{t+1})u'(c_{t+1}^o) = u'(c_t^y)$$

Substitute in the budget constraints for both ages:

$$\beta(1 + r_{t+1})u'([1 + r_{t+1}]s_{t+1}) = u'(w_t - s_{t+1})$$

This implicitly defines a **saving function**

$$s_{t+1} = s(w_t, r_{t+1}) \tag{3}$$

Properties of the saving function

Totally differentiate the Euler Equation to find the derivatives of the saving function:

$$\begin{aligned} & \beta(1+r_{t+1})^2 u''([1+r_{t+1}]s_{t+1}) ds_{t+1} \\ = & u''(w_t - s_{t+1})(dw_t - ds_{t+1}) \end{aligned}$$

Effect of higher endowments:

$$\frac{ds_{t+1}}{dw_t} = \frac{u''(c_t^y)}{\beta(1+r_{t+1})^2 u''(c_{t+1}^o) + u''(c_t^y)} > 0$$

Simply an income effect.

Saving function

Effect of the interest rate

$$\begin{aligned} & \beta u'([1 + r_{t+1}]s_{t+1})dr_{t+1} \\ & + \beta(1 + r_{t+1})u''(c_{t+1}^o)(s_{t+1}dr_{t+1} + [1 + r_{t+1}]ds_{t+1}) \\ = & -u''(w_t - s_{t+1})ds_{t+1} \end{aligned}$$

Saving function

Effect of the interest rate

The effect is ambiguous (income vs substitution effects):

$$\frac{\partial s_{t+1}}{\partial r_{t+1}} = - \frac{\beta u'(c_{t+1}^o) + \beta(1+r_{t+1})u''(c_{t+1}^o)s_{t+1}}{\beta(1+r_{t+1})^2 u''(c_{t+1}^o) + u''(c_t^y)} \quad (4)$$

Use the 2nd period budget constraint to replace $(1+r_{t+1})s_{t+1}$ by c_{t+1}^o .

$$\frac{\partial s_{t+1}}{\partial r_{t+1}} = - \frac{\beta u'(c_{t+1}^o) + \beta u''(c_{t+1}^o) c_{t+1}^o}{\beta(1+r_{t+1})^2 u''(c_{t+1}^o) + u''(c_t^y)} \quad (5)$$

Saving function

Effect of the interest rate

Define the coefficient of relative risk aversion:

$$\sigma(c) \equiv -u''(c)c/u'(c) \quad (6)$$

Then

$$\frac{\partial s_{t+1}}{\partial r_{t+1}} = -\frac{\beta u'(c_{t+1}^o)(1 - \sigma[c_{t+1}^o])}{\beta(1 + r_{t+1})^2 u''(c_{t+1}^o) + u''(c_t^y)} \quad (7)$$

It follows that savings respond positively to the interest rate, if $\sigma > 1$.

- High $\sigma \rightarrow$ small substitution effect \rightarrow income effect raises c_t^y / reduces s_{t+1} .

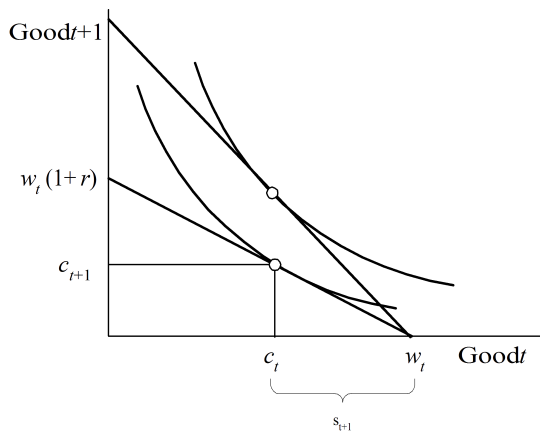
- In particular, in the popular CRRA utility function

$$u(c) = c^{1-\sigma} / (1 - \sigma)$$

the coefficient is constant (namely σ , show this!).

- For $\sigma = 1$, this becomes log utility (and $s_r = 0$).
- In the data, σ is most likely greater than one, although its value is highly controversial.

Effect of a higher interest rate



The figure illustrates the case where income and substitution effect just cancel.

Law of motion for capital

Recall $(1 + n)k_{t+1} = s(w_t, r_{t+1})$.

Use the firm FOCs to replace the prices:

$$(1 + n)k_{t+1} = s(f(k_t) - f'(k_t)k_t, f'(k_{t+1}))$$

This is a first order difference equation of the form

$$k_{t+1} = \phi(k_t)$$

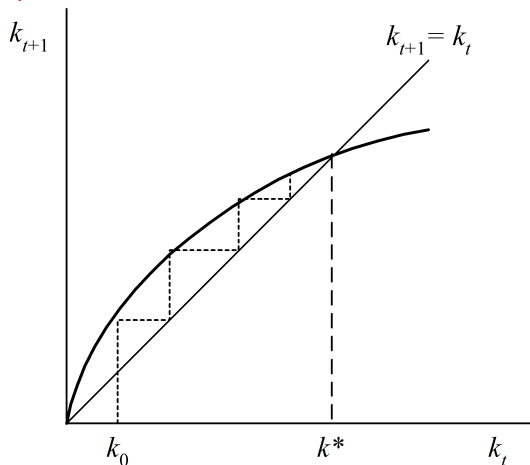
Implicitly differentiating yields

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_w k_t f''(k_t)}{1 + n - s_r f''(k_{t+1})} \quad (8)$$

This completely determines the behavior of the economy.

Concave law of motion

If ϕ is concave, we get simple dynamics.



From any initial condition (k_0) the economy converges monotonically to a unique steady state (k^*).

Properties of the law of motion

We know:

- $\phi(0) = 0$: $k = 0$ is a steady state.
- The derivative is

$$\frac{dk_{t+1}}{dk_t} = \frac{-s_w k_t f''(k_t)}{1 + n - s_r f''(k_{t+1})} \quad (9)$$

- A sufficient condition for $\phi' > 0$ is $s_r > 0$. Intuition: the supply of capital is upward sloping.

Otherwise, little can be said in general.

Log utility - Cobb Douglas example

The utility function is $u(c) = \ln(c)$.

Then the household saves a constant fraction of his earnings:

$$c_t^y = w_t / (1 + \beta)$$

and therefore

$$s_{t+1} = w_t \beta / (1 + \beta)$$

Log utility - Cobb Douglas example

Assume further that $f(k) = k^\theta$. Then

$$w = (1 - \theta)k^\theta$$

The law of motion then becomes

$$(1 + n)k_{t+1} = \frac{\beta}{1 + \beta}(1 - \theta)k_t^\theta$$

Because $s_r = 0$ and s_w is a constant, ϕ inherits the curvature of the production function.

A unique, stable steady state exists.

Log utility - Cobb Douglas example

Steady state

$$k^* = \left[\frac{1 - \theta}{1 + n} \frac{\beta}{1 + \beta} \right]^{1/(1-\theta)}$$

Steady state interest rate:

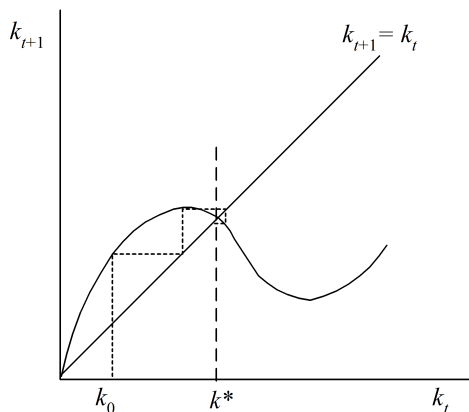
$$\begin{aligned} f'(k) &= \theta k^{\theta-1} \\ f'(k^*) &= \frac{\theta}{1 - \theta} \frac{1 + \beta}{\beta} (1 + n) \\ r &= f'(k) - \delta \end{aligned}$$

Note: the steady state interest rate could be very small (low θ or high β) or very large.

Log utility - Cobb Douglas example

- The example provides a microfoundation for the Solow model.
- But it is a special case.

An ill behaved example



The economy oscillates towards the steady state.

Multiple steady states are possible.

An important insight: Even very simple models can have surprisingly complicated (and unpleasant) dynamics.

Definition

A steady state is an equilibrium where all (per capita) variables are constant.

Output is bounded

What is the maximum sustainable level of $f(k)$?

Set consumption to 0 and find the steady state:

$$\begin{aligned}(1+n)k &= f(k) + (1-\delta)k \\ \Rightarrow \\ f(k) &= (n+\delta)k\end{aligned}$$

The Golden Rule

What is the maximum sustainable consumption level?

Consumption per young household is

$$c^y + c^o / (1 + n) = f(k) + (1 - \delta)k - (1 + n)k'$$

Impose the steady state requirement $k' = k$ and maximize with respect to k :

$$f'(k_{GR}) = n + \delta \quad (10)$$

Intuition ...

$k > k_{GR}$ implies a Pareto inefficient allocation.

- By running down the capital stock, households at all dates could eat more.

Nothing rules out a steady state that is dynamically inefficient.

Dynamic Inefficiency

- Why is this possible?
- There is a missing market: the old must finance their consumption out of own saving, even if the rate of return is very low.
- If a young household saves one unit of consumption, he has $f'(k') + 1 - \delta$ units of consumption at old age.
- If all generations entered into a contract where today's young transfer one unit of consumption (per young household) to the current old, each old household would receive $1 + n$ units.
- Dynamic inefficiency means over accumulation to the point where the rate of return of saving falls below the “rate of return” of this transfer scheme.
 - We will return to this idea in the section on “social security.”

Why does the First Welfare Theorem fail?

- Vaguely, the First Welfare Theorem says: when all markets are competitive and some other conditions hold, every CE is Pareto Optimal.
- One of the "other conditions" comes in 2 flavors:
 - 1 there is a finite number of goods
 - 2 $\sum_{j=1}^{\infty} p_j < \infty$ where p_j are the CE (Arrow-Debreu) prices.
- Both conditions are violated in the OLG model.
- Acemoglu, ch. 9.1.

- Acemoglu, ch. 9.
- Krueger, "Macroeconomic Theory," ch. 8
- Sargent & Ljungqvist, ch. 9 (without the monetary parts).
- McCandless & Wallace and De la Croix and Michelle are book-length treatments of overlapping generations models.