Problem Set 1: OLG Models

Econ720. Prof. Lutz Hendricks. September 8, 2011

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1 An Economy with Land

Consider a two-period OLG model in which production requires land and labor.

Demographics: Each period a cohort of size $N = 1$ is born. Each person lives for 2 periods.

Preferences: Households value consumption when young and old according to

$$u(c_t^y) + \beta u(c_{t+1}^o)$$

Endowments: At time 0, the old each hold M units of land. Each young person supplies one unit of labor.

Technology: Output is produced from land and labor according to the production function $F(M, L)$. F has constant returns to scale.

Markets: Households rent labor to firms at wage w_t. Households rent land to firms at rental price r_{t+1}. Land is traded at price q_t.

Questions:

1. Write down the household's budget constraints. Note that the household sells his land holdings at price q_{t+1} when old.
2. Derive the household's FOCs and Euler equation.
3. State and solve the firm's problem.
4. Define a competitive equilibrium.
5. Derive an implicit solution for q in steady state.

2 OLG Model with Assets

Demographics: There are two types of households, indexed by h. In each period, a mass of 0.5 households is born of each type. Each person lives for 2 periods.

Endowments: Households receive endowments (e^y, e^o) when young and old, respectively.

Preferences: $\ln\left(c_{h,t}^y\right) + \beta_h \ln\left(c_{h,t+1}^o\right)$.

Technologies: None.

Markets: Households trade goods and one period bonds that are issued and purchased by households.

Questions:

1. Define a solution to the household problem. Solve for the household's bond supply function.
2. Solve for the equilibrium bond interest rate.
3. Your solution for R should reveal the following features: (i) If old endowments are larger, R is higher. (ii) If β_h increases, R decreases. (iii) R is time invariant. Provide intuition for these features.
4. Now add a durable good to the economy. It is in fixed supply, K. It pays a dividend d per period. Households trade "shares" of this good in an asset market at price p_t, measured in units of consumption goods. Define a competitive equilibrium for this economy.
5. Why do you find that the number of equations equals the number of objects to be determined? Usually, we find that we have one additional equation, which is redundant by Walras' law.
6. Derive an equation that determines the equilibrium price sequence p_t.

3 Answers: PS1

3.1 Answer: An Economy With Land

Life is much easier when solving this model, if you distinguish household m from firm m from aggregate M. Otherwise you have to be very careful with market clearing conditions.

(a) The budget constraints are

$$w_t = c_t^y + q_t m_t$$

and

$$c_{t+1}^o = (q_{t+1} + r_{t+1}) m_t$$

(b) The household problem is therefore

$$\max u\left(w_t - q_t m_t\right) + \beta u\left((q_{t+1} + r_{t+1}) m_t\right)$$

The Euler equation is

$$u'(c_t^y) q_t = \beta u'(c_{t+1}^o)(q_{t+1} + r_{t+1})$$

$$R_{t+1} = (q_{t+1} + r_{t+1})/q_t$$

The Euler equation can then be written as

$$u'(c_t^y) = \beta u'(c_{t+1}^o) R_{t+1}$$

which looks just like the equation from a model with bonds or capital. It implicitly defines a savings function giving m_t as a function of prices (after substituting for consumption). The solution to the household problem is thus a triple

$$(c_t^y, c_{t+1}^o, m_t)$$

that satisfies the 2 budget constraints and the Euler equation.

(c) The FOCs for the firms are as usual:

$$r_{t+1} = f'(m_t)$$

and

$$w_t = f(m_t) - f'(m_t) m_t$$

(d) There are four markets: goods, land (rental and purchase), and labor. Goods market clearing requires

$$c_t^y + c_t^o = f(M)$$

Land rental and labor market clearing are implicit in the notation. The market for land clears if savings equal supply of land or $m = M$.
A CE is then a sequence of prices (w, q, r) and quantities (c^y, c^o, m, L) that satisfy the household's Euler equation and budget constraints (3 equations), the firm's FOCs (2 equations), and the 3 market clearing conditions. We thus have 8 equations and 7 unknowns. Convince yourself that the two household budget constraints imply the goods market clearing condition; so we are fine.
(e) In equilibrium, the Euler equation becomes

$$q_t u'(w_t - q_t M) = \beta u'((q_{t+1} + r_{t+1})M)(q_{t+1} + r_{t+1})$$

With log utility this can be solved in closed form:

$$q/[w - qM] = \beta/M \tag{1}$$

Therefore, $w/q - M = M/\beta \Rightarrow$

$$qM = \frac{\beta w(M)}{1+\beta}$$

3.2 Answer Sketch: OLG Model with Assets

1. Euler equation: $c_{h,t+1}^o / c_{h,t}^y = \beta_h R_{t+1}$. Budget constraint: $W_t = e^y + e^o/R_{t+1} = c_{h,t}^y + c_{h,t+1}^o / R_{t+1}$. Young consumption function: $c_{h,t}^y = W/[1+\beta]$. Bond supply from the young budget constraint:

$$b_{h,t+1} = e^y - c_{h,t}^y = e^y - [e^y + e^o/R_{t+1}]/[1+\beta_h]$$

   Solution: $\left(c_{h,t}^y, c_{h,t+1}^o, b_{h,t+1}\right)$ that satisfy 2 policy functions and one budget constraint.

2. Bond market clearing: $\sum_h b_{h,t+1} = 0$.

$$\begin{aligned} 2e^y &= [e^y + e^o/R] \sum_h [1+\beta_h]^{-1} \\ R &= \frac{e^o}{e^y} \left[\frac{2}{\sum_h (1+\beta_h)^{-1}} - 1\right]^{-1} \end{aligned}$$

3. Intuition: (i) and (ii) High old endowments or low β: agents want to save less. But aggregate saving is fixed at 0. Need to adjust interest rate. (iii) Essentially a consequence of lack of intergenerational trade.

4. Nothing changes in the household problem, except that household asset holdings are $s_{h,t+1} = b_{h,t+1} + p_t k_{h,t+1}$ where $R_{t+1} = (d + p_{t+1})/p_t$ by no arbitrage. Goods market clearing requires $e^y + e^o + Kd = 0.5 \sum_h \left(c_{h,t}^y + c_{h,t+1}^o\right)$. Asset market clearing requires $\sum_h b_{h,t+1} = 0$ and $0.5 \sum_h k_{h,t+1} = K$. A CE consists of sequences $c_{h,t}^y, c_{h,t+1}^o, b_{h,t+1}, k_{h,t+1}, s_{h,t+1}, R_{t+1}, p_t$ (12 objects) which satisfy:

- Household: 2 policy functions and 1 budget constraint per type (6 equations)
- Market clearing: 3 equations
- Definition of s: 2 equations
- Definition of R: 1 equation.

5. We have only 12 equations because the model does not determine portfolio compositions for either household type.

6. Capital market clearing requires:

$$2p_t K = \sum p_t k_{h,t+1} = \sum p_t k_{h,t+1} + \sum b_{h,t+1} = \sum s_{h,t+1}$$

because from bond market clearing, aggregate saving equals aggregate wealth. As in the model without K, $s_{h,t+1} = e^y - c^y_{h,t+1} = e^y - W/(1 + \beta_h)$. Therefore

$$2pK = 2e^y - \left[e^y + e^o \frac{p}{p+d} \right] \sum (1 + \beta_h)^{-1}$$