

# Perpetual youth

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- The standard growth model is very tractable.
- But it has an important limitation: all households are identical.
- For some questions, it is important to have households of **different ages**:
  - fiscal policies that redistribute across ages
  - models with life-cycle features: job search, matching, ...
- An analytically tractable version of the OLG model is the Blanchard-Yaari model of perpetual youth.

- At  $t = 0$ , there are  $L(0) = 1$  identical persons.
  - They are all newborns.
- At each instant,  $nL(t)$  identical persons are born.
- Each person dies at each instant with **Poisson** probability  $\nu$ .
- The population growth rate is  $n - \nu > 0$ :

$$L(t) = \exp([n - \nu] t) \quad (1)$$

- The Poisson process is the continuous time analog of i.i.d.
- It is a counting process: it describes the distribution of the number of events occurring during a particular time interval.
- It is a one-parameter distribution, characterized by the arrival rate  $\nu$ .
- The probability of no even over a period of length  $\tau$  is  $\exp(-\nu\tau)$ .
- At each instant, fraction  $\nu$  of the population experiences an event.

- The mass of persons at  $t$  aged  $t - \tau$  is

$$\begin{aligned}L(t|\tau) &= n \exp(-\nu(t - \tau) + (n - \nu)\tau) \\ &= \Pr(\text{live beyond } t - \tau) nL(\tau)\end{aligned}$$

- Notation:  $x(t|\tau)$  means  $x$  at  $t$  for those born at  $\tau$ .

- Households are indexed by  $i$ .
- Conditional on surviving, households utility at date  $t$  is  $e^{-\rho t} \ln(c_i(t))$ .
- The probability of being alive after  $t$  "periods" is  $\exp(-vt)$ .
- Expected utility for date  $t$  is  $e^{-vt} e^{-\rho t} \ln(c_i(t))$ .
- Expected lifetime utility is

$$\int_0^{\infty} e^{-(\rho+v)t} \ln(c_i(t)) dt \quad (2)$$

- Interesting: mortality simply increases the discount factor:  $\rho + v$ .

- The resource constraint is

$$\dot{K} + C = F(K, L) - \delta K$$

- In per capita terms

$$\dot{k} = f(k) - c - (n - v + \delta)k \quad (3)$$

- $k = K/L$  is capital per capita and capital per worker.

- A representative firm solves the standard problem.
- Factor prices are

$$R = f'(k)$$

$$w = f(k) - f'(k)k$$

- The representative member of cohort  $t - \tau$  solves

$$\max \int_{t-\tau}^{\infty} e^{-(\rho+\nu)t} \ln(c(t|\tau)) dt$$

subject to

$$\dot{a}(t|\tau) = r(t)a(t|\tau) - c(t|\tau) + w(t) + z(a(t|\tau)|t, \tau) \quad (4)$$

- This is standard, except for the transfers  $z$ .

# Accidental Bequests

- Households die holding assets.
- What to do with these "accidental bequests"?
- We introduce "life insurance."

# Accidental Bequests

- Assumption: households buy fair **annuities**.
- Give  $a(t|\tau)$  to the insurance company.
- Get paid:
  - 1 interest  $r(t)a(t|\tau)$
  - 2 an equal share of accidental bequests of your own cohort:

$$z(a(t|\tau) | t, \tau) = va(t|\tau) \quad (5)$$

- Effectively, the interest rate, conditional on survival, is  $r(t) + v$ .

$$\dot{a}(t|\tau) = [r(t) + v]a(t|\tau) - c(t|\tau) + w(t) \quad (6)$$

## Definition

A CE is an allocation  $[K(t), c(t|\tau), a(t|\tau)]_{t=0, \tau \leq t}^{\infty}$  and a price system  $[w(t), R(t)]$  such that:

1.  $c(t|\tau)$  and  $a(t|\tau)$  solve the household's problem for cohort  $t - \tau$ .
2.  $w(t)$  and  $R(t)$  solve the firm's problem.
3. markets clear.

Market clearing:

- labor: implicit
- capital:  $K(t) = \int_{-\infty}^t L(t|\tau) a(t|\tau) d\tau$ .
- goods: same as resource constraint.

Identity:  $r(t) = R(t) - \delta$ .

- This is a standard problem with Euler equation

$$\frac{\dot{c}(t|\tau)}{c(t|\tau)} = r(t) - \rho \quad (7)$$

budget constraint and TVC

$$\lim_{t \rightarrow \infty} \exp(-(\bar{r}(t, \tau) + \nu)[t - \tau]) a(t|\tau) = 0 \quad (8)$$

- $\bar{r}$  is the average interest rate

$$\bar{r}(t, \tau) = \frac{1}{t - \tau} \int_{\tau}^t r(s) ds \quad (9)$$

- Claim: because of log utility, the household consumes a constant fraction of "wealth:"

$$c(t|\tau) = (\rho + \nu) [a(t|\tau) + \omega(t)] \quad (10)$$

- Human wealth for all alive at  $t$  is the same:

$$\omega(t) = \int_t^\infty \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) w(s) ds \quad (11)$$

- Because all households have the same wealth and consume the same fraction of it:

$$c(t) = \int_{-\infty}^t L(t, \tau) c(t|\tau) d\tau / L(t) \quad (12)$$

$$= (\rho + \nu) [a(t) + \omega(t)] \quad (13)$$

- This is a form of **aggregation**: Aggregate consumption behaves like individual consumption.
  - As if a single individual made the choice.
- The budget constraint aggregates in the same way.
- How general is this?

- We have a system in  $c, a, \omega$ .
- Equations: consumption function, budget constraint, def of lifetime wealth:

$$c(t) = (\rho + \nu) [a(t) + \omega(t)]$$

$$\dot{a}(t) = (r(t) - (n - \nu)) a(t) + w(t) - c(t)$$

$$\omega(t) = \int_t^\infty \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) w(s) ds$$

- Differentiate the consumption function:

$$\dot{c} = (\rho + \nu) [\dot{a} + \dot{\omega}] \quad (14)$$

- Sub in budget constraint for  $\dot{a}$ .
- Differentiate def of  $\omega$  (Leibniz's rule - next slide):

$$\dot{\omega}(t) = (r(t) + \nu) \omega(t) - w(t) \quad (15)$$

- Sub that into  $\dot{c}$  and collect terms:

$$\dot{c}(t) = [r(t) - \rho] c(t) - (\rho + \nu) n a(t) \quad (16)$$

- Sub in  $k(t) = a(t)$  and the firm foc for  $r(t)$ :

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + \nu) n \frac{k(t)}{c(t)} \quad (17)$$

## Note: Differentiating $w(t)$

$$\omega(t) = \int_t^{\infty} \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) w(s) ds \quad (18)$$

$\dot{\omega}(t)$  has 2 pieces:

- 1 Effect of changing lower bound of integral is integrand evaluated at  $s = t$ :  $w(t)$ .
- 2 Derivative of integrand w.r.to  $t$ :  
 $-[r(t) + \nu] \omega(t) = \int_t^{\infty} w(s) \frac{d}{dt} \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) ds.$

Now note that

$$\frac{d}{dt} \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) = \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) \times [-(r(t) + \nu)].$$

# Intuition for $w(t)$

- Think of human wealth as an asset with price  $w(t)$ .
- Its instantaneous payoff consists of:
  - 1 "dividend"  $w(t)$
  - 2 capital gain  $\dot{w}(t)$
- The asset price equals [required rate of return]  $\times$  [dividend + capital gain]
- Required rate of return is  $r(t) + v$ .

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + \nu) n \frac{k(t)}{c(t)} \quad (19)$$

$$\dot{k} = f(k) - c - (n - \delta - \nu) k \quad (20)$$

with boundary conditions  $k(0)$  given and TVC (which is not so obvious...)  
This looks a lot like a standard growth model...

$$\dot{c} = 0 \implies$$

$$c = \frac{(\rho + \nu) n}{f'(k) - \delta - \rho} k \quad (21)$$

Properties:

- 1  $k \longrightarrow 0 \implies c \longrightarrow 0$  [as  $f' \longrightarrow \infty$ ]
- 2  $k \longrightarrow k^{GR}$  where  $f'(k^{GR}) = \delta + \rho \implies c \longrightarrow \infty$
- 3  $c''(k) > 0$  [verify]

$$\dot{k} = 0 \implies$$

$$c = f(k) - (n + \delta - v)k \quad (22)$$

Properties: as the standard growth model.

Solution for steady state  $k^*$

$$\frac{f(k^*)}{k^*} - (n - v + \delta) - \frac{(\rho + v)n}{f'(k^*) - \delta - \rho} = 0 \quad (23)$$

Unique steady state  $k^*$ :  $f(k)/k \searrow$  in  $k$ .  $-1/f'(k) \searrow$  in  $k$ .

- **Golden Rule** maximizes

$$c^* = f(k^*) - (n + \delta - \nu)k^* \quad (24)$$

$$f'(k_{GR}) = (n + \delta - \nu) \quad (25)$$

- Steady state:

$$f'(k^*) > \rho + \delta \quad (26)$$

[otherwise  $c/k < 0$ ]

- There can be overaccumulation relative to the Golden Rule.
- This happens when households are sufficiently impatient (high  $\rho$ ).
- Similar to the finite lifetime OLG model.

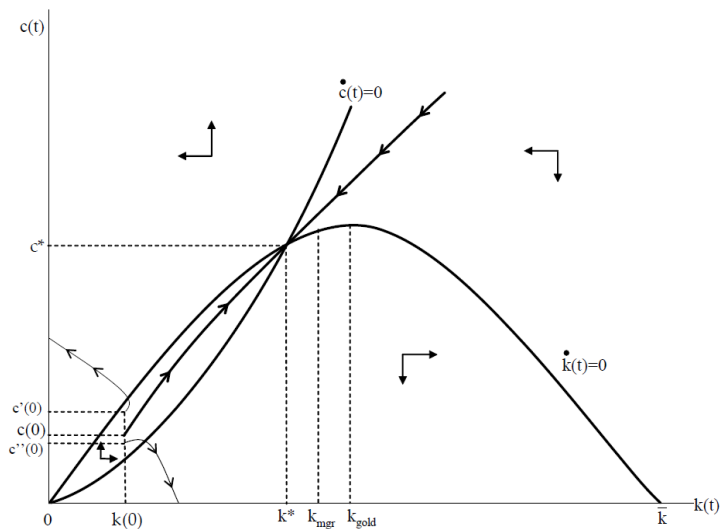
- **Modified Golden Rule** for planner with discount factor  $\rho$  [effects of mortality and "annuities" cancel]:

$$f'(k_{MGR}) = \rho + \delta \quad (27)$$

- Equilibrium avoids overaccumulation relative to MGR.
- This is not a robust feature of the model.
- Giving households a stronger motive to save for "old age" can lead to overaccumulation.
- Example: labor efficiency declines with age.

- Finite lifetimes are not necessary to generate overaccumulation.
- In this model, it is the presence of overlapping generations that destroys the welfare theorems.

# Phase diagram



# Phase diagram

- The dynamics closely resemble the growth model.
- A unique, globally saddle path stable steady state exists.
- Convergence is monotone.
- An analytically tractable model with OLG.

- Acemoglu, Introduction to modern economic growth, ch. 9.7-9.8.