

Perpetual youth

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- The standard growth model is very tractable.
- But it has an important limitation: all households are identical.
- For some questions, it is important to have households of **different ages**:
 - fiscal policies that redistribute across ages
 - models with life-cycle features: job search, matching, ...
- An analytically tractable version of the OLG model is the Blanchard-Yaari model of perpetual youth.

- At $t = 0$, there are $L(0) = 1$ identical persons.
- At each instant, $nL(t)$ identical persons are born.
- Each person dies at each instant with **Poisson** probability ν .
- The population growth rate is $n - \nu > 0$:

$$L(t) = \exp([n - \nu] t) \quad (1)$$

- The Poisson process is the continuous time analog of i.i.d.
- It is a counting process: it describes the distribution of the number of events occurring during a particular time interval.
- It is a one-parameter distribution, characterized by the arrival rate ν .
- The probability of no even over a period of length τ is $\exp(-\nu\tau)$.
- At each instant, fraction ν of the population experiences an event.

- The mass of persons aged $t - \tau$ is

$$\begin{aligned}L(t|\tau) &= \exp(-\nu(t - \tau) + (n - \nu)t) \\ &= \Pr(\text{live beyond } t - \tau) L(\tau)\end{aligned}$$

- Notation: $x(t|\tau)$ means x at t for those born at τ .

- Households are indexed by i .
- Conditional on surviving, households utility at date t is $e^{-\rho t} \ln(c_i(t))$.
- The probability of being alive after t "periods" is $\exp(-vt)$.
- Expected utility for date t is $e^{-vt} e^{-\rho t} \ln(c_i(t))$.
- Expected lifetime utility is

$$\int_0^{\infty} e^{-(\rho+v)t} \ln(c_i(t)) dt \quad (2)$$

- The resource constraint is

$$\dot{K} + C = F(K, L) - \delta K$$

- In per capita terms

$$\dot{k} = f(k) - c - (n - v + \delta)k \quad (3)$$

- A representative firm solves the standard problem.
- Factor prices are

$$R = f'(k)$$

$$w = f(k) - f'(k)k$$

- The representative member of cohort $t - \tau$ solves

$$\max \int_{t-\tau}^{\infty} e^{-(\rho+\nu)t} \ln(c(t|\tau)) dt$$

subject to

$$\dot{a}(t|\tau) = r(t)a(t|\tau) - c(t|\tau) + w(t) + z(a(t|\tau)|t, \tau) \quad (4)$$

- This is standard, except for the transfers z .

- The issue is what to do with accidental bequests.
- Assumption: households face fair annuities markets.
- Give $a(t|\tau)$ to the insurance company.
- Get paid:
 - 1 interest $r(t) a(t|\tau)$
 - 2 an equal share of accidental bequests of your own cohort:

$$z(a(t|\tau) | t, \tau) = \nu a(t|\tau) \quad (5)$$

- Effectively, the interest rate, conditional on survival, is $r(t) + \nu$.

$$\dot{a}(t|\tau) = [r(t) + \nu]a(t|\tau) - c(t|\tau) + w(t) \quad (6)$$

Definition

A CE is an allocation $[K(t), c(t|\tau), a(t|\tau)]_{t=0, \tau \leq t}^{\infty}$ and a price system $[w(t), R(t)]$ such that:

1. $c(t|\tau)$ and $a(t|\tau)$ solve the household's problem for cohort $t - \tau$.
2. $w(t)$ and $R(t)$ solve the firm's problem.
3. markets clear.

Market clearing:

- labor: implicit
- capital: $K(t) = \int_{-\infty}^t L(t|\tau) a(t|\tau) d\tau$.
- goods: same as resource constraint.

Identity: $r(t) = R(t) - \delta$.

- This is a standard problem with Euler equation

$$\frac{\dot{c}(t|\tau)}{c(t|\tau)} = r(t) - \rho \quad (7)$$

budget constraint and TVC

$$\lim_{t \rightarrow \infty} \exp(-(\bar{r}(t, \tau) + \nu)[t - \tau]) a(t|\tau) = 0 \quad (8)$$

- \bar{r} is the average interest rate

$$\bar{r}(t, \tau) = \frac{1}{t - \tau} \int_{\tau}^t r(s) ds \quad (9)$$

- Claim: because of log utility, the household consumes a constant fraction of "wealth:"

$$c(t|\tau) = (\rho + \nu) [a(t|\tau) + \omega(t)] \quad (10)$$

- Human wealth for all alive at t is the same:

$$\omega(t) = \int_t^\infty \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) w(s) ds \quad (11)$$

- Because all households have the same wealth and consume the same fraction of it:

$$c(t) = (\rho + \nu) [a(t) + \omega(t)] \quad (12)$$

- We have a system in c, a, ω .
- Equations: consumption function, budget constraint, def of lifetime wealth:

$$c(t) = (\rho + \nu) [a(t) + \omega(t)]$$

$$\dot{a}(t) = (r(t) - (n - \nu)) a(t) + w(t) - c(t)$$

$$\omega(t) = \int_t^\infty \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) w(s) ds$$

- Differentiate the consumption function:

$$\dot{c} = (\rho + \nu) [\dot{a} + \dot{\omega}] \quad (13)$$

- Sub in budget constraint for \dot{a} .
- Differentiate def of ω :

$$\dot{\omega}(t) = (r(t) + \nu) \omega(t) - w(t) \quad (14)$$

- Sub that into \dot{c} and collect terms:

$$\dot{c}(t) = [r(t) - \rho] c(t) - (\rho + \nu) n a(t) \quad (15)$$

- Sub in $k(t) = a(t)$ and the firm foc for $r(t)$:

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + \nu) n \frac{k(t)}{c(t)} \quad (16)$$

Note: Differentiating $w(t)$

$$\omega(t) = \int_t^{\infty} \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) w(s) ds \quad (17)$$

$\dot{\omega}(t)$ has 2 pieces:

- 1 Effect of changing lower bound of integral is integrand evaluated at $s = t$: $w(t)$.
- 2 Derivative of integrand w.r.to t :
 $-[r(t) + \nu] \omega(t) = \int_t^{\infty} w(s) \frac{d}{dt} \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) ds.$

Now note that

$$\frac{d}{dt} \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) = \exp\left(-\int_t^s [r(\iota) + \nu] d\iota\right) \times [-(r(t) + \nu)].$$

Intuition for $w(t)$

- Think of human wealth as an asset with price $w(t)$.
- Its instantaneous payoff consists of:
 - 1 "dividend" $w(t)$
 - 2 capital gain $\dot{w}(t)$
- The asset price equals [required rate of return] \times [dividend + capital gain]
- Required rate of return is $r(t) + v$.

$$\frac{\dot{c}(t)}{c(t)} = f'(k(t)) - \delta - \rho - (\rho + \nu) n \frac{k(t)}{c(t)} \quad (18)$$

$$\dot{k} = f(k) - c - (n - \delta - \nu) k \quad (19)$$

with boundary conditions $k(0)$ given and TVC (which is not so obvious...)
This looks a lot like a standard growth model...

$$\dot{c} = 0 \implies$$

$$c = \frac{(\rho + \nu) n}{f'(k) - \delta - \rho} k \quad (20)$$

Properties:

- 1 $k \longrightarrow 0 \implies c \longrightarrow 0$ [as $f' \longrightarrow \infty$]
- 2 $k \longrightarrow k^{GR}$ where $f'(k^{GR}) = \delta + \rho \implies c \longrightarrow \infty$
- 3 $c''(k) > 0$ [verify]

$$\dot{k} = 0 \implies$$

$$c = f(k) - (n + \delta - \nu)k \quad (21)$$

Properties: as the standard growth model.

Solution for steady state k^*

$$\frac{f(k^*)}{k^*} - (n - v + \delta) - \frac{(\rho + v)n}{f'(k^*) - \delta - \rho} = 0 \quad (22)$$

Unique steady state k^* : $f(k)/k \searrow$ in k . $-1/f'(k) \searrow$ in k .

- **Golden Rule** maximizes

$$c^* = f(k^*) - (n + \delta - \nu)k^* \quad (23)$$

$$f'(k_{GR}) = (n + \delta - \nu) \quad (24)$$

- Steady state:

$$f'(k^*) > \rho + \delta \quad (25)$$

[otherwise $c/k < 0$]

- There can be overaccumulation relative to the Golden Rule.
- This happens when households are sufficiently impatient (high ρ).
- Similar to the finite lifetime OLG model.

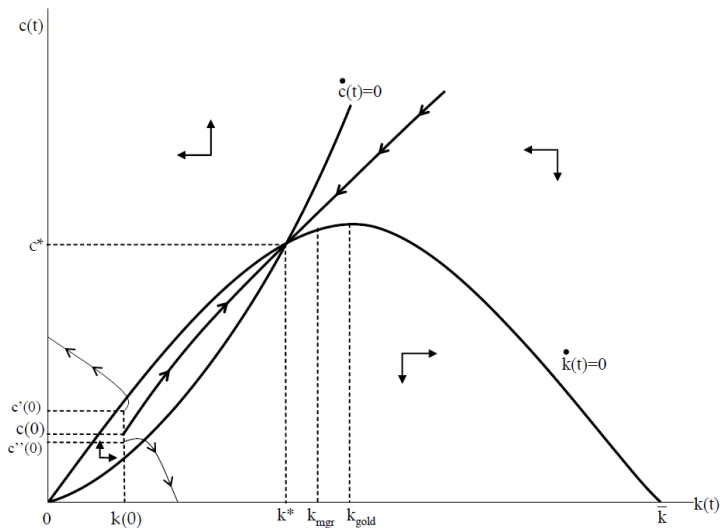
- **Modified Golden Rule** for planner with discount factor ρ [effects of mortality and "annuities" cancel]:

$$f'(k_{MGR}) = \rho + \delta \quad (26)$$

- Equilibrium avoids overaccumulation relative to MGR.
- This is not a robust feature of the model.
- Giving households a stronger motive to save for "old age" can lead to overaccumulation.
- Example: labor efficiency declines with age.

- Finite lifetimes are not necessary to generate overaccumulation.
- In this model, it is the presence of overlapping generations that destroys the welfare theorems.

Phase diagram



Phase diagram

- The dynamics closely resemble the growth model.
- A unique, globally saddle path stable steady state exists.
- Convergence is monotone.
- An analytically tractable model with OLG.

- Acemoglu, Introduction to modern economic growth, ch. 9.7-9.8.