# Optimal Control

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# Topics

- Optimal control is a method for solving dynamic optimization problems in continuous time.

# Generic Optimal control problem

Choose functions of time $c(t)$ and $k(t)$ so as to

$$\max \int_0^T v[k(t), c(t), t]dt \tag{1}$$

Constraints:

1. Law of motion of the **state** variable $k(t)$:

$$\dot{k}(t) = g[k(t), c(t), t] \tag{2}$$

2. Feasible set for **control** variable $c(t)$:

$$c(t) \in Y(t) \tag{3}$$

3. Boundary conditions:

$$k(0) = k_0, \text{given} \tag{4}$$

$$k(T) \geq k_T \tag{5}$$

# Generic Optimal control problem

- $c$ and $k$ can be vectors.
- $Y(t)$ is a compact, nonempty set.
- $T$ could be infinite. Then $\lim_{t \to \infty} k(t) \geq k_T$.
- Important: the state cannot jump; the control can.

# Example

A household chooses optimal consumption to

$$\max \int_0^T u[c(t)]dt \tag{6}$$

subject to

$$\dot{k}(t) = rk(t) - c(t) \tag{7}$$

$$c(t) \in [0, \bar{c}] \tag{8}$$

$$k(0) = k_0, \text{given} \tag{9}$$

$$k(T) \geq 0 \tag{10}$$

# A Recipe for Solving Optimal Control Problems

# A Recipe

1. Write down the **Hamiltonian**

$$H(t) = v(k, c, t) + \mu(t) g(k, c, t) \tag{11}$$

- $\mu$ is essentially a Lagrange multiplier (called a "**co-state** variable).

# A Recipe

2. Derive the **first order conditions** which are **necessary** for an optimum:

$$\partial H/\partial c = 0 \quad (12)$$

$$\partial H/\partial k = -\dot{\mu} \quad (13)$$

# A Recipe

3. Impose the **transversality** condition:

- for finite horizon:

$$\mu(T) = 0 \tag{14}$$

- for infinite horizon:

$$\lim_{t \to \infty} H(t) = 0 \tag{15}$$

- this depends on the terminal condition (see below).

# A Recipe

4. A **solution** is the a set of functions $[c(t), k(t), \mu(t)]$ which satisfy
   - the FOCs
   - the law of motion for the state
   - the boundary / transversality conditions

# Details

- First order conditions are necessary, not sufficient.
- They are necessary only if we **assume** that
  1. a continuous, interior solution exists;
  2. the objective function $v$ and the constraint function $g$ are continuously differentiable.
- Acemoglu 7 offers some insight into why the FOCs are necessary.

# Details

- If there are multiple states and controls, simply write down one FOC for each separately:

$$\begin{aligned} \delta H / \delta c_i &= 0 \\ \partial H / \partial k_j &= -\dot{\mu}_j \end{aligned}$$

- There is a large variety of cases depending on the length of the horizon (finite or infinite) and the kinds of boundary conditions.
    - Each has its transversality condition (see Leonard & Van Long).

# Next steps

Typical useful next things to do:

1. Eliminate $\mu$ from the system. Obtain two differential equations in $(c, k)$ .
2. Find the steady state by imposing $\dot{c} = \dot{k} = 0$.

# Sufficient conditions

First-order conditions are sufficient, if the programming problem is **concave**.

- The objective function and the constraints are concave functions of the controls and the states.
- The co-state must be positive.
- This condition is easy to check, but very stringent.

# Sufficient conditions

First-order conditions are sufficient, if the Hamiltonian is concave in controls and states, where the co-state is evaluated at the optimal level.

- This, too is very stringent.

# Sufficient conditions

- Arrow-Kurz 1970: First-order conditions are sufficient, if the *maximized* Hamiltonian is concave in the states.
- The maximized Hamiltonian is defined as follows.
  1. Maximize the Hamiltonian by choosing controls.
  2. Substitute those controls into the Hamiltonian.
  3. Then substitute the co-states that satisfy the FOCs into the Hamiltonian.
- This is less stringent and by far the most useful set of sufficient conditions.

# Discounting: Current value Hamiltonian

- The generic discounted problem is

$$\max \int_0^T e^{-\rho t} v[k(t), c(t)] dt \tag{16}$$

subject to the same constraints as above.
- The only change: utility depends on $t$ only through $e^{-\rho t}$.
- Of course, the undiscounted solution method still works.

# Current value Hamiltonian

- A more convenient method: use a "**current value Hamiltonian**."
- This does not change the Hamiltonian or the FOC.
- The only change is the derivative with respect to the co-state:

$$\delta H / \delta k = \mu(t)\rho - \dot{\mu}(t) \tag{17}$$

and the TVC

$$\lim_{T \to \infty} e^{-\rho T} \mu(T) k(T) = 0 \tag{18}$$

# Current value Hamiltonian

In the background: The original Hamiltonian is

$$\hat{H} = \max e^{-\rho t} v(k, c) + \hat{\mu} g(k, c, t) \tag{19}$$

One can always multiply by a constant, $e^{-\rho t}$:

$$\begin{aligned} H &= \max v(k, c) + \mu g(k, c, t) \\ \mu &= \hat{\mu} e^{\rho t} \end{aligned}$$

Apply the recipe for the standard Hamiltonian to find the recipe for the current value Hamiltonian.

# Equality constraints

Equality constraints of the form

$$h[c(t), k(t), t] = 0 \tag{20}$$

are simply added to the Hamiltonian as in a Lagrangian problem:

$$H(t) = v(k, c, t) + \mu(t) g(k, c, t) + \lambda(t) h(k, c, t) \tag{21}$$

FOCs are unchanged:

$$\begin{aligned} \partial H / \partial c &= 0 \\ \partial H / \partial k &= -\dot{\mu} \end{aligned}$$

# Transversality Conditions

# Finite horizon: Scrap value problems

- The horizon is $T$.
- The objective function assigns a scrap value to the terminal state variable: $e^{-\rho T}\phi(k(T))$:

$$\max \int_0^T e^{-\rho t} v[k(t), c(t), t] dt + e^{-\rho T}\phi(k(T)) \tag{22}$$

- Hamiltonian and FOCs: unchanged.
- Replace the TVC with

$$\mu(T) = \phi'(k(T)) \tag{23}$$

- Intuition: $\mu$ is the marginal value of the state $k$.

# Infinite horizon TVC

- The finite horizon TVC with the boundary condition $k(T) \geq k_T$ is $\mu(T) = 0$.
    - Intuition: capital has no value at the end of time.
- But the infinite horizon boundary condition is NOT $\lim_{t\to\infty} \mu(t) = 0$.
- The next example illustrates why.

# Infinite horizon TVC: Example

$$\max \int_0^{\infty} \left[\ln\left(c\left(t\right)\right) - \ln\left(c^*\right)\right] dt$$

$$\textit{subject to}$$

$$\begin{aligned} \dot{k}\left(t\right) &= k\left(t\right)^{\alpha} - c\left(t\right) - \delta k\left(t\right) \\ k\left(0\right) &= 1 \\ \lim_{t\to\infty} k\left(t\right) &\geq 0 \end{aligned}$$

where $c^*$ is the max steady state (golden rule) consumption.

# Infinite horizon TVC

- Hamiltonian

$$H(k, c, \lambda) = \ln c - \ln c^* + \lambda [k^\alpha - c - \delta k] \tag{24}$$

- Necessary FOCs

$$H_c = 1/c - \lambda = 0 \tag{25}$$
$$H_k = \lambda [\alpha k^{\alpha-1} - \delta] = -\dot{\lambda} \tag{26}$$

# Infinite horizon TVC

- We show: $\lim_{t\to\infty} c(t) = c^*$ [why?]
- Limiting steady state solves

$$\begin{aligned} \dot{\lambda}/\lambda &= \alpha k^{\alpha-1} - \delta = 0 \\ \dot{k} &= k^{\alpha} - 1/\lambda - \delta k = 0 \end{aligned}$$

- Solution is the golden rule:

$$k^* = (\alpha/\delta)^{1/(1-\alpha)} \tag{27}$$

- Verify that this max's steady state consumption.

# Infinite horizon TVC

- Implications for the TVC...
- $\lambda(t) = 1/c(t)$ implies $\lim_{t\to\infty} \lambda(t) = 1/c^*$.
- Therefore, neither $\lambda(t)$ nor $\lambda(t)\, k(t)$ converge to 0.
- The correct TVC: $\lim_{t\to\infty} H(t) = 0$.
- The only reason why the standard TVC does not work: there is no discounting in the example.

# Infinite horizon TVC: Discounting

- With discounting, the TVC is easier to check.
- Assume:
  - the objective function is $e^{-\rho t} v \left[k(t), c(t)\right]$
  - it only depends on $t$ through the discount factor
  - $v$ and $g$ are weakly monotone
- Then the TVC becomes

$$\lim_{t \to \infty} e^{-\rho t} \mu(t) k(t) = 0 \tag{28}$$

where $\mu$ is the costate of the current value Hamiltonian.

- This is exactly analogous to the discrete time version

$$\lim_{t \to \infty} \beta^t u'(c_t) k_t = 0 \tag{29}$$

# Example: renewable resource

# Example: Renewable resource

$$\begin{aligned}
& \max \int_0^\infty e^{-\rho t} u\left(y\left(t\right)\right) dt && (30) \\
& subject\ to && (31) \\
& \dot{x}\left(t\right) - y\left(t\right) && (32) \\
x\left(0\right) &= 1 && (33) \\
x\left(t\right) &\geq 0 && (34)
\end{aligned}$$

# Example: Renewable resource

Current value Hamiltonian

$$H(x, y, \mu) = u(y) - \mu y \tag{35}$$

Necessary FOCs

$$u'(y) = \mu \tag{36}$$

$$\dot{\mu} = \rho\mu \tag{37}$$

- $\dot{\mu} = \rho\mu \implies \mu(t) = \mu(0)\, e^{\rho t}$.
- $u'(y) = \mu \implies$

$$y(t) = u'^{-1}\left[\mu(0)\, e^{\rho t}\right] \tag{38}$$

- The optimal path has $\lim x(t) = 0$ or

$$\int_0^\infty u'^{-1}\left[\mu(0)\, e^{\rho t}\right] dt = 1 \tag{39}$$

- This solves for $\mu(0)$.

# Example: Renewable resource
## TVC

- TVC for infinite horizon case:

$$\lim e^{-\rho t}\mu(0)\,e^{\rho t}x(t) = 0 \qquad (40)$$

- Equivalent to

$$\lim x(t) = 0 \qquad (41)$$

# Application to the growth model

# Application to the growth model

$$\max \int_0^\infty e^{-\rho t} u\left(c\left(t\right)\right) dt \tag{42}$$

subject to

$$\begin{aligned} \dot{k}\left(t\right) &= f\left(k\left(t\right)\right) - c\left(t\right) - \delta k\left(t\right) && (43) \\ & k\left(0\right) \text{ given} && (44) \end{aligned}$$

# Current value Hamiltonian

$$H(k, c, \mu) = u(c(t)) + \mu(t)[f(k(t)) - c(t) - \delta k(t)] \quad (45)$$

Necessary conditions:

$$\begin{aligned} H_c &= u'(c) - \mu = 0 \\ H_k &= \mu\left[f'(k) - \delta\right] = \rho\mu - \dot{\mu} \end{aligned}$$

# TVC

- $u$ is monotone.
- $g(k,c) = f(k) - c - \delta k$ is monotone in $c$ but not $k$.
- However, we "know" that $k$ never rises above the golden rule point where $f'(k) = \delta$ - unless $k(0)$ is too high.
- Then $g$ is increasing in $k$.
- The TVC becomes:

$$\lim_{t\to\infty} e^{-\rho t}\mu(t)\,k(t) = 0 \tag{46}$$

# Sufficiency

- This is an example where the easiest (1st) set of sufficiency conditions applies:
- $u$ is strictly concave in $c$ (only).
- $g(k, c)$ is jointly concave in $k$ and $c$.
- First order conditions are sufficient.

# Reading

- Acemoglu, "Introduction to Modern Economic Growth," ch. 7. Proves the Theorems of Optimal Control.
- Barro & Sala-i-Martin, appendix.
- Sundaram (1996).
- Leonard, Daniel; Ngo van Long (1992). *Optimal control theory and static optimization in economics*. Cambridge: Cambridge University Press. A fairly comprehensive treatment. Contains many variations on boundary conditions.