

The Growth Model In Continuous Time

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- 1 Tools:
 - 1 Solving models in continuous time (optimal control).
 - 2 Phase diagrams.
- 2 Solow model.
- 3 Cass-Koopmans / neoclassical growth model.

[Some of you will find the next several slides obvious.]

Continuous Time vs. Discrete Time

- So far, time was divided into discrete "periods."
- It is often more convenient to shrink the length of periods to 0.
- Difference equations then become differential equations.

Continuous time

Example: Law of motion for capital

- Discrete time:

$$K_{t+1} - K_t = I_t - \delta K_t \quad (1)$$

- More generally:

$$K_{t+\Delta t} - K_t = [I_t - \delta K_t] \Delta t \quad (2)$$

- Continuous time ($\Delta t \rightarrow 0$):

$$\lim_{\Delta t \rightarrow 0} \frac{K_{t+\Delta t} - K_t}{\Delta t} = \dot{K}_t = I_t - \delta K_t \quad (3)$$

Growth rates in continuous time

The growth rate of a variable is defined as

$$g(x) = \frac{\dot{x}}{x} = \frac{d \ln x}{dt} \quad (4)$$

Growth rate rules:

- 1 $g(xy) = g(x) + g(y)$.
- 2 $g(x/y) = g(x) - g(y)$.
- 3 $g(x^\alpha) = \alpha g(x)$.
- 4 $x(t) = e^{\gamma t} \implies g(x) = \gamma$.

Differential equations

Differential equations

Take a function of time:

$$x(t) = a + bt \quad (5)$$

There is another way of describing this function:

- Take the derivative:

$$\dot{x}(t) = dx(t) / dt = b \quad (6)$$

- Fix $x(0) = a$.
- The two pieces of information (the derivative and $x(0)$) completely describe $x(t)$.
- Only one function $x(t)$ satisfies both pieces.
 - But note that infinitely many functions satisfy the derivative!

Definition: Differential equation

- A differential equation (DE) is a function of the form

$$\dot{x}(t) = f(x(t), t) \quad (7)$$

- This is actually a "first-order" DE.
- **Higher order** DEs contain higher order derivatives of time.
 - E.g.: A second order DE

$$d^2x(t) / dt^2 + dx(t) / dt = a + bt \quad (8)$$

- Together with a boundary condition, the DE can be solved for $x(t)$.

The bad news: There is no algorithm for solving DEs.

- But one look up solutions in tables.
- It is also easy to **verify** a solution one may guess.
- Or one can integrate the DE.

The good news: The DEs we will use are all very similar.

- We will make a list of cases and their solutions.

Consider again

$$\dot{x}(t) = b \quad (9)$$

$$x(0) = a \quad (10)$$

Guess

$$x(t) = a + bt \quad (11)$$

Verify:

- Take the time derivative and find that it matches $\dot{x} = b$.
- Verify that $x(0) = a$.

Example

$$\dot{x}(t) = b x(t) \quad (12)$$

$$x(0) = a \quad (13)$$

Guess:

$$x(t) = a e^{bt} \quad (14)$$

Verify: Take the derivative

$$\dot{x}(t) = b a e^{bt} = b x(t) \quad (15)$$

$$x(0) = a e^0 = a \quad (16)$$

Example: Boundary conditions I

$$\dot{x}(t) = b x(t) \quad (17)$$

$$x(T) = a \quad (18)$$

Guess:

$$x(t) = D e^{bt} \quad (19)$$

Verify:

$$\dot{x}(t) = b D e^{bt} = b x(t) \quad (20)$$

Find D from

$$x(T) = a = D e^{bT} \quad (21)$$

Example: Boundary conditions II

Boundary conditions can take many forms:

- $\int_a^b x(s) ds = 5.$
- $\dot{x}(T) = 5.$
- $x(T) - x(T - 2) = 5.$
- etc.

The Solow Model

The Solow Model - Structure

- Modify the discrete time growth model in two ways:
 - 1 Continuous time.
 - 2 Fixed saving rate.
- This is not an equilibrium model, but can be interpreted as one.

- Firms as the same as in the Ramsey model.
- They rent labor L and capital K from households.
- The production function has constant returns to scale.
- The firm's problem

$$\max F(K, L) - wL - qK \quad (22)$$

- Define $k^F = K/L$ and

$$f(k^F) = F(K, L) / L \quad (23)$$

- The first order conditions are then

$$q = f'(k^F) \quad (24)$$

and

$$w = f(k^F) - f'(k^F)k^F \quad (25)$$

- Population size:

$$L_t = e^{nt} \quad (26)$$

- Labor supply is also L_t .
- Preferences are

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \quad (27)$$

- Budget constraint

$$\dot{K}_t = w_t L_t + (r_t - \delta) K_t - L_t c_t$$

Households

Per capita budget constraint

- Define $k = K/L$.
- Law of motion for k :

$$\begin{aligned}\dot{k}/k &= \dot{K}/K - n \\ &= w/k + (r - \delta) - c/k\end{aligned}$$

- Or

$$\dot{k}_t = w_t + (r_t - \delta - n)k_t - c_t \quad (28)$$

Households

Constant saving rate

- The modern way: Set up an optimization problem and derive the saving function.
- The Solow way: Assume that the saving rate is fixed:

$$c = (1 - s)(w + rk) \quad (29)$$

- Therefore:

$$\dot{k} = s(w + rk) - (n + \delta)k \quad (30)$$

Factor market clearing requires

$$k = k^F \quad (31)$$

Goods market clearing requires

$$F(K_t, L_t) = C_t + \delta K_t + \dot{K}_t$$

or in per capita terms

$$\dot{k} = f(k) - (n + \delta)k - c \quad (32)$$

An equilibrium is a collection of *functions* (of time)

$$c_t, k_t, k_t^F, w_t, r_t, q_t$$

that satisfy

- 1 the firm's first order conditions (2)
- 2 the household's budget constraint and the behavioral equation

$$\dot{k} = s(w + rk) - (n + \delta)k$$

- 3 market clearing (2)
- 4 the identity $r = q - \delta$.

- The entire model boils down to to one key equation:

$$\dot{k}_t = sf(k_t) - (n + \delta)k_t \quad (33)$$

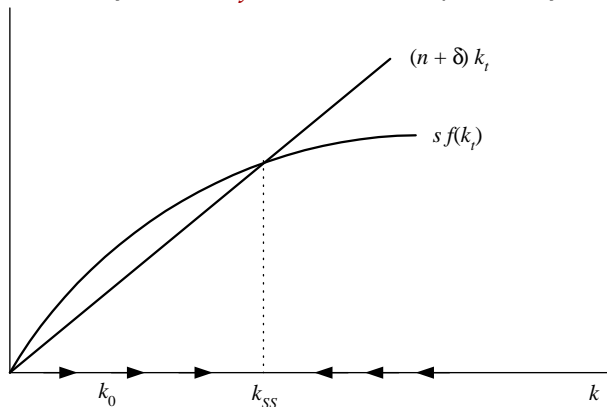
- This is simply the household's behavioral equation after applying $f(k) = w + rk$.

Steady state

The steady state requires $\dot{k} = 0$ or

$$sf(k) = (n + \delta)k \quad (34)$$

With strictly concave f , there is a unique steady state with $k > 0$.



Steady state: Golden Rule

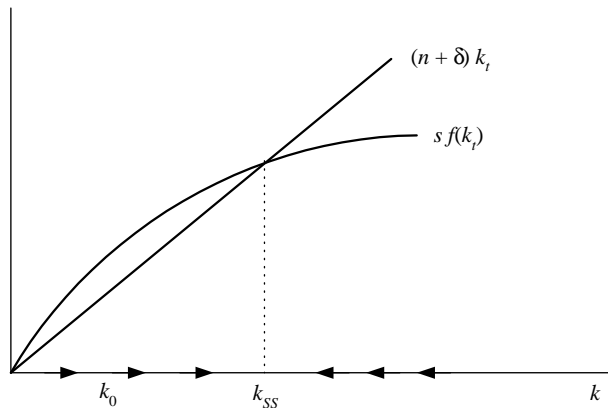
- Which k maximizes steady state consumption?
- Steady state consumption is

$$c = f(k) - (n + \delta)k \quad (35)$$

- Maximizing this yields the first-order condition:

$$f'(k^{GR}) - \delta = n \quad (36)$$

- This is the exact analogue to the discrete time case: The interest rate must equal the population growth rate.



The steady state is *stable*.
Convergence is monotone.

Adding Technical Change

- The model does not have sustained growth in per capita income.
- This requires technical change (A grows).
- Assume exogenous growth in A :

$$A(t) = A(0)e^{\gamma t} \quad (37)$$

Adding Technical Change

- Assume that technical change takes the following form:

$$Y(t) = F(K(t), A(t)L(t)) \quad (38)$$

- This type of technical change is called "labor-augmenting" or "Hicks-neutral".
- This is the *only* form of technical change that is consistent with *balanced growth*.

Definition

A balanced growth path is a path along which all growth rates are constant.

How to analyze a growing model?

- Construct a **stationary transformation**.
- Divide each variable by its balanced growth factor:

$$\tilde{x}(t) = x(t) e^{-g_x t} \quad (39)$$

where g_x is the **balanced growth rate** of x .

- Or take ratios of variables that grow at the same rate.
- The economy in transformed variables (\tilde{x}) has a steady state.

How to find the balanced growth rates?

- For equations that involve sums:

$$Y(t) = C(t) + I(t) + G(t) \quad (40)$$

Constant growth (usually) requires that all summands grow at the same rate.

- For other equations: Try taking the growth rate of the whole equation.
- Example:

$$Y(t) = K(t)^\alpha [A(t)L(t)]^{1-\alpha} \quad (41)$$

implies

$$g(Y) = \alpha g(K) + (1 - \alpha) [g(A) + n] \quad (42)$$

Balanced growth path: Solow Model

- Start from

$$\dot{K}(t) = sF(K(t), A(t)L(t)) - \delta K(t) \quad (43)$$

$$g(K(t)) = sF\left(1, \frac{A(t)L(t)}{K(t)}\right) - \delta \quad (44)$$

- Constant growth requires that

$$\bar{k}(t) = \frac{K(t)}{A(t)L(t)} \quad (45)$$

be constant over time. Thus, on a balanced growth path:

$$g(K) = \gamma + n \quad (46)$$

- Production function:

$$\bar{y}(t) = \frac{Y(t)}{A(t)L(t)} = F(\bar{k}(t), 1) \quad (47)$$

must be constant on a balanced growth path.

- Thus: The model has a steady state in (\bar{k}, \bar{y}) .

$$\begin{aligned}g(\bar{k}) &= g(K) - \gamma - n \\ &= sF\left(1, \frac{A(t)L(t)}{K(t)}\right) - \delta - \gamma - n \\ &= sf(\bar{k})/\bar{k} - \delta - \gamma - n\end{aligned}$$

Or

$$\dot{\bar{k}}(t) = sf(\bar{k}(t)) - (n + \delta + \gamma)\bar{k}(t) \quad (48)$$

Nothing changes, except the constant term in the law of motion.

- Acemoglu, "Introduction to Modern Economic Growth," ch. 2 covers the Solow model and stationary transformations of growing economies.
- Barro & Sala-i-Martin, ch. 1
- Romer, Advanced Macroeconomics, ch. 1