Comparative Dynamics

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Comparative Dynamics

- We use phase diagrams to uncover the dynamic response to shocks.
- We study tax changes in a growth model.

Model

The **household** solves

$$\max \int_0^\infty e^{-\rho t} u(c_t)\, dt \tag{1}$$

subject to

$$\dot{k}_t = r_t k_t + w_t - c_t - \tau_t \tag{2}$$

and k_0 given.
Firms produce output using $F(K, L)$.
The **government** uses the tax revenue to finance goverment spending: $G_t = \tau_t$.

Competitive Equilibrium

A competitive equilibrium consists of functions $c(t), k(t), \tau(t), w(t), r(t)$ that satisfy:

1. Household: Budget constraint and
$$g(c) = \frac{r-\rho}{\sigma} \tag{3}$$

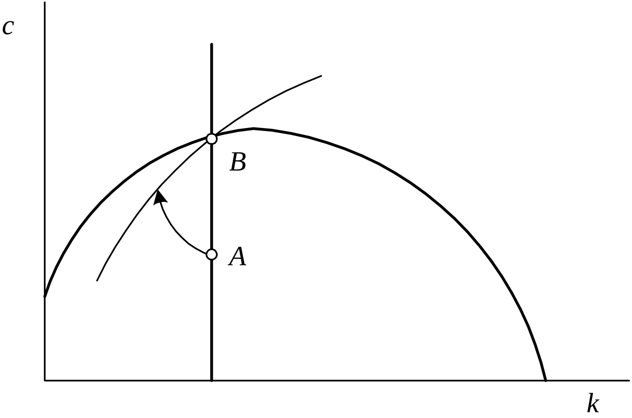
2. Firms:
$$\begin{aligned} r &= f'(k) - \delta && (4) \\ w &= f(k) - f'(k)\,k && (5) \end{aligned}$$

3. Government:
$$\tau = G \tag{6}$$

4. Market clearing:
$$\dot{k} = f(k) - \delta k - c - G \tag{7}$$

Phase Diagram

The only change relative to the model without government: G shifts the $\dot{k} = 0$ locus down.



Permanent Tax Increase

Consider a permanent, unannounced increase in G.
In the phase diagram

- $\dot{k} = 0$ locus shifts down by ΔG.
- k_{ss} remains unchanged because the $\dot{c} = 0$ locus does not shift.

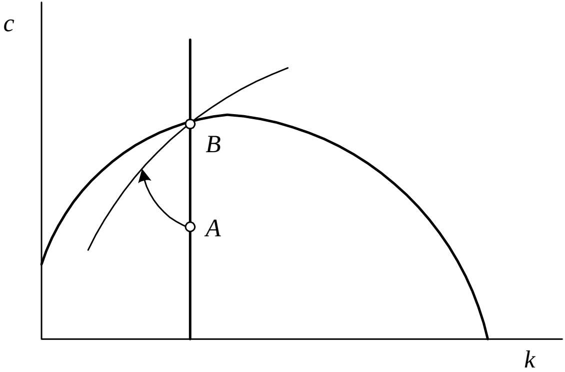
Dynamics: c_{ss} drops to the new saddle path, then moves along it.
An interesting long-run result: full crowding out of consumption $(\Delta c_{ss} = -\Delta G)$.

Temporary Tax Increase

- Consider a *temporary*, unannounced increase in G.
    - $G_t = G^* + \Delta G$ for $0 \leq t \leq T$, but $G_t = G^*$ for $t > T$.
- To find the dynamics, we work backwards.
- Start from $t = T$: the economy looks like one without taxes (on saddle path).
- Consider $0 < t < T$:
    - The phase diagram with taxes applies.
    - But the economy is not on the saddle path (why not?).
    - Key point: consumption cannot jump, except when new info arrives.
- We need to construct a path that follows the with-tax phase diagram and connects with the no tax saddle path at $t = T$.

Temporary Tax Increase

- What happens at date 0? The $\dot{k} = 0$ locus shifts down.
- Consumption must drop; otherwise the TVC would be violated (the economy would forever move North-West).
- Consider $k_0 = k_{ss}$. One candidate path: c_0 drops by less than ΔG.



Temporary Tax Increase

- Consider $k_0 < k_{ss}$.
- c_0 drops.
- It cannot drop below the new saddle-path because then it would not reach the old saddle-path at T.
- The economy must move north-west and reach the old saddle-path at T.

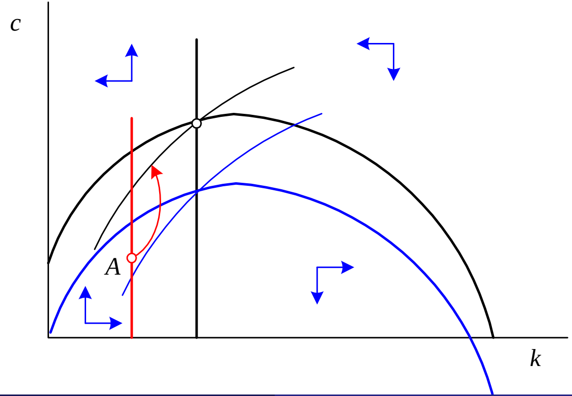
Announced Tax Cut

Consider a surprise tax cut that is announced to take place at date T.

At $t = 0$ the news arrives that taxes remain low until $t = T$, but then fall permanently.

We need to construct a path that connects with the low-tax saddle path at $t = T$.

Before T, the dynamics is governed by the high tax phase diagram



Phase Diagram for a Simple Human Capital Model

A Human Capital Model

We study the decision of a household how much human capital to accumulate.
This example illustrates two complications:

1. finite horizons
2. binding inequality constraints.

Household problem

- The household who lives for an interval of length T.
- Human capital is accumulated according to

$$\dot{h}(t) = v(t) - \delta\, h(t) \tag{8}$$

where v is time spent accumulating human capital.

- The household maximizes

$$\int_0^T e^{-\rho\, t}\, u(c(t))\, dt \tag{9}$$

- subject to (8) and the budget constraint

$$c(t) = w(t)\, h(t)\, [1 - \tau(t)\, v(t)] \tag{10}$$

- The restriction on time is $v \geq 0$.
- For simplicity, assume that $v \leq 1$ never binds.

Household: Intuition

- Human capital acquired early is more valuable for two reasons:
  1. it lives longer (date T is farther off);
  2. its payoffs are discounted by less.
- We expect the optimal path for $v(t)$ to be falling over time.
- When close to T, we expect $v(t) \geq 0$ to bind.

Hamiltonian

$$H = u\left(wh\left[1 - \tau\, v\right]\right) + \lambda\, \left[v - \delta\, h\right] \tag{11}$$

First-order conditions

$$u'(c)\, wh\tau \geq \lambda \tag{12}$$

with equality if $v > 0$ and

$$\dot{\lambda} = \rho\lambda - u'(c)\, w\left(1 - \tau\, v\right) + \lambda\, \delta \tag{13}$$

Summary

The solution to the household problem consists of functions (c, h, v, λ) that solve

1. The first-order conditions

$$\begin{aligned} u'(c)\,wh\tau &\geq \lambda && (14)\\ \dot{\lambda} &= (\rho+\delta)\lambda - u'(c)\,w\,(1-\tau\,v) && (15) \end{aligned}$$

   with equality if $v > 0$.
2. The budget constraint

$$c(t) = w(t)\,h(t)\,[1-\tau(t)\,v(t)] \qquad (16)$$

3. The law of motion

$$\dot{h}(t) = v(t) - \delta\,h(t) \qquad (17)$$

4. The boundary conditions: h_0 given and $\lambda_T = 0$.

Log utility

With log utility, the optimality conditions simplify:

$$\begin{aligned}
\lambda &\leq \tau \text{ if } v = 0 \\
\lambda &= \frac{\tau}{1 - \tau v} \text{ if } v > 0 \\
\dot{\lambda} &= (\rho + \delta)\,\lambda - 1/h \\
\dot{h} &= v - \delta h \\
\lambda_T &= 0
\end{aligned}$$

Steady State

- Assume that w and τ are constant over time and that $T = \infty$.
- Then h and v converge to stationary levels, h_{ss} and v_{ss}.
- We next determine those levels.
- $\dot{\lambda} = 0$ implies

$$(\rho + \delta)\, h\lambda = (\rho + \delta)\, h \frac{\tau}{1 - \tau v} = 1 \tag{18}$$

- $\dot{h} = 0$ implies

$$v = \delta h \tag{19}$$

- Combine both

$$h_{ss} = [\tau\, (\rho + 2\,\delta)]^{-1} \tag{20}$$

Steady State

It follows that

$$\begin{aligned} v_{ss} &= \delta\, h_{ss} = \frac{\delta}{\tau[\rho + 2\,\delta]} \\ c_{ss} &= \frac{(\rho + \delta)\, w}{(\rho + 2\,\delta)^2\, \tau} \end{aligned}$$

$$\lambda_{ss} = u'\left(\frac{(\rho + \delta)\, w}{(\rho + 2\,\delta)^2\, \tau}\right) \frac{w}{\rho + 2\,\delta}$$

Phase Diagram

- What is the optimal path of v over time?
- Assume log utility.
- The phase diagram has two regions: $v = 0$ and $v > 0$.
- The region boundary occurs when the household just hits the constraint $v \geq 0$: at $\lambda = \tau$.
- For $\lambda > \tau$: $v > 0$.
- For $\lambda \leq \tau$: $v = 0$.

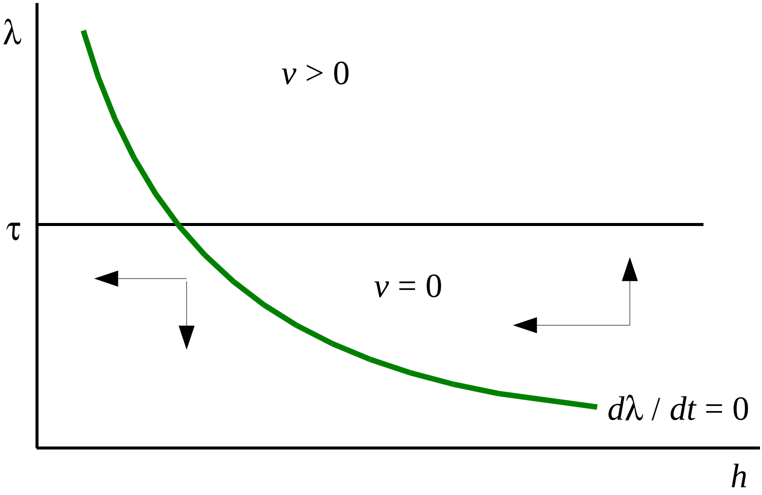
Region v = 0

- The shadow price λ is not large enough to cover the opportunity cost τ.
- The household does not invest in human capital.
- The laws of motion are:

$$\begin{aligned}\dot{\lambda} &= (\rho+\delta)\lambda - 1/h \\ \dot{h} &= -\delta h\end{aligned}$$

- $\lambda \uparrow \Rightarrow \dot{\lambda} \uparrow$.
- $h \uparrow \Rightarrow \dot{\lambda} \uparrow$ and $\dot{h} \downarrow$.
- Hence, $h(t) = h(t_0)\, e^{-\delta\,(t-t_0)}$, where t_0 is any date at which the economy is inside the region.

Phase Diagram: Region v = 0



Phase Diagram: Region v>0

The first-order condition for v holds with equality:

$$\lambda\left(1-\tau v\right)=\tau$$

or

$$v=1/\tau-1/\lambda \tag{21}$$

Substitute v out of the law of motion:

$$\dot{h}=1/\tau-1/\lambda-\delta h \tag{22}$$

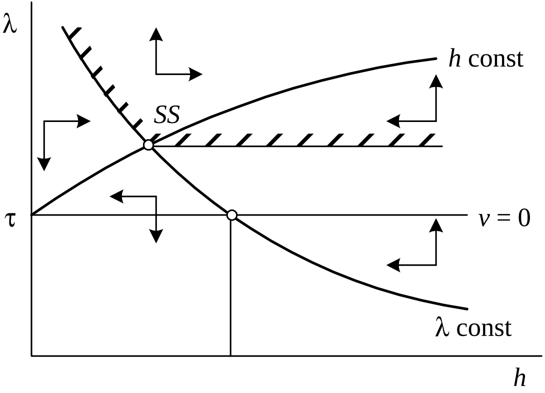
Keep

$$\dot{\lambda}=(\rho+\delta)\lambda-1/h \tag{23}$$

Phase Diagram: Region v$>$0

- In this region, the shadow price of human capital (λ) equals the opportunity cost.
- $\dot{h} = 0$ is upward sloping and starts at $\lambda = \tau$.
- $\dot{\lambda} = 0$ is a downward sloping hyperbola (as in region $v = 0$).
- $h \uparrow$ or $\lambda \downarrow \Rightarrow \dot{h} \downarrow$.

Phase Diagram



Dynamics

- Any path must end with $\lambda_T = 0$ exactly at date T.
- It follows that the shaded region must never be entered.
- What happens as the steady state is approached with $v > 0$?
  - Since all the laws of motion are continuous, $\dot{h} \to 0$ and $\dot{\lambda} \to 0$.
  - The steady state can never be reached.
  - But the economy can spend an arbitrarily long time arbitrarily close to the steady state.

Dynamics

- First consider $h_0 < h_{ss}$.
- λ depends on the horizon T.
- Short T: λ is low.
    - Start in region $v = 0$
    - move south-west until $\lambda_T = 0$.

Dynamics

- To prove this, solve the two differential equations.
- $h(t) = h(t_0)\, e^{-\delta t}$. Substitute this into the law of motion for λ to obtain

$$\dot{\lambda} = (\rho + \delta)\, \lambda - e^{\delta t} / h(t_0) \tag{24}$$

- The solution to this differential equation is

$$\lambda(t) = e^{(\rho+\delta)\,(t-t_0)} \left[\lambda(t_0) - \frac{\rho}{h(t_0)} \left\{ 1 - e^{-\rho(t-t_0)} \right\} \right]$$

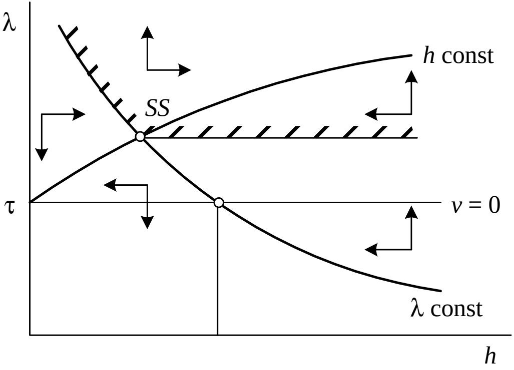
- Imposing the boundary condition $\lambda(T) = 0$ implies $\lambda(t_0)\, h(t_0) = \rho \left\{ 1 - e^{-\rho(T-t_0)} \right\}$.
- For long T: $\lambda(t_0) \rightarrow \rho / h(t_0)$ (unless the region $v = 0$ is left).
- But for a short T, $\lambda(t_0) \rightarrow 0$.

Dynamics

- Case: $h_0 < h_{ss}$ and long T.
- Initially $v > 0$ and the economy moves south until it crosses into the $v = 0$ region.
- As $T \rightarrow \infty$ something bizarre happens:
  - the economy approaches the steady state without ever reaching it.
  - It comes arbitrarily close and stays arbitrarily close for an arbitrarily long time.
  - But when the terminal date comes sufficiently close it leaves the steady state and moves south-west to reach $\lambda_T = 0$.

Dynamics

- Case $h_0 > h_{ss}$.
- Investment is never large enough to increase h.
- The economy may move straight south-west if T is short or it may move towards the steady state, similar to the case where $h_0 < h_{ss}$.



Read

- Acemoglu, Introduction to modern economic growth, ch. 8.7.
- Hendricks, Lutz (2004). "Taxation and Human Capital Accumulation." *Macroeconomic Dynamics* 8(3): 310-334.
- Sheshinski, Eytan (1968), "On the Individual's Lifetime Allocation Between Education and Work," *Metroeconomica*, 20(1), 42-9.