Problem Set 5: Growth Model in Continuous Time

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1 Capital income tax

[Romer 2.9] Consider a Ramsey economy on its balanced growth path. At time 0 the government starts to tax capital income, so that the interest rate facing the household is

$$r(t) = (1-\tau)\ f'(k_t)$$

where I have assumed that $\delta = 0$. Tax revenues are rebated to the household in a lump-sum fashion. The change in the policy is unanticipated.

1. How does the tax affect the $\dot{k} = 0$ and the $\dot{c} = 0$ loci?
2. How do the balanced growth values of c and k change?
3. Describe the changes at time 0 and the transition path thereafter.
4. Show that the saving rate on the balanced growth path ($[y-c]/y$) is decreasing in τ.
5. Imagine there are two countries that differ only in τ. Do the residents of the high τ countries have an incentive to invest in the low τ country or vice versa?
6. How do your answers change if the tax revenues are used to pay for government purchases instead of being rebated?

2 Continuous Time CIA Model. Cash and Credit Goods.

Crusoe solves the following problem:

$$\max \int_0^\infty e^{-\rho t}\, u(c_t, g_t)\ dt$$

subject to the budget constraint

$$\dot{k}_t + c_t + g_t + \dot{M}_t/p_t = f(k_t) + x_t$$

and the CIA constraint

$$c_t \leq M_t/p_t$$

The notation is standard. There are two consumption goods, which are perfect substitutes in production but not in consumption. The cash good c is subject to the CIA constraint, while the credit good g is not. Denote real balances by $m_t = M_t/p_t$. x_t is a money transfer from the government.

(a) Write down the household's Hamiltonian. Which are his states and controls? Derive first-order conditions for two cases: either the CIA constraint always binds or it never binds.

(b) Define a competitive equilibrium. Assume that the government lets the money stock grow at the constant rate $g(M)$.

(c) Derive a set of equations that characterize the steady state. Show that the nominal interest rate equals zero, if the CIA constraint does not bind.

(d) Determine the effects of a higher money growth rate on the steady state allocation. Assume that the utility function takes the form $u(c,g) = U(c) + V(g)$, where U and V are strictly concave functions.

3 Answers PS5

3.1 Answer: Capital income tax

1. Nothing happens to the law of motion for k because the tax revenues are rebated. The law of motion for c changes to

$$\dot{c}/c = \frac{(1-\tau)\ f'(k) - \delta - \rho - \theta g(A)}{\theta} \tag{1}$$

Therefore $\dot{c} = 0$ implies

$$(1-\tau) f'(k) = \delta + \rho + \theta g(A) \tag{2}$$

This shifts left as τ rises (need lower k to maintain the same after tax interest rate).

2. Balanced growth Draw the phase diagram to see that both k and c fall.

3. Transition. At time 0 c jumps up or down to the new saddle path. Thereafter c and k both converge monotonically.

4. BGP saving rate. The BGP saving rate is

$$s = \frac{y-c}{y} = (n + \delta + g)\frac{k}{y}$$

We already showed that

$$\tau \uparrow \Rightarrow k \downarrow \Rightarrow k/y \downarrow \Rightarrow s \downarrow \tag{3}$$

5. Pre-tax interest rates are higher in high tax countries. If capital income is taxed in the country of residence, then low tax residents want to invest in high tax countries.

6. With government spending the law of motion for capital becomes

$$\dot{k} = f(k) - \delta k - c - \tau f'(k) - (g(A) + n) k$$

The $\dot{k} = 0$ locus shifts down as τ rises (and gets distorted). This does not change the steady state change in k, but lowers the initial and steady state c.

3.2 Answer: Continuous Time CIA Model. Cash and Credit Goods.

(a) The state is $a = m + k$. The controls are c, g, m. The Hamiltonian is

$$H = u(c,g) + \lambda\ [f(a-m) + x - c - g - \pi\, m] + \varphi\ [m - c]$$

where $\pi = \dot{p}/p$ is the inflation rate. First-order condtions are:

$$\begin{aligned} u_c &= \lambda + \varphi \\ u_g &= \lambda \\ \lambda\,(f'(a-m) + \pi) &= \varphi \\ \dot{\lambda} &= \lambda\,(\rho - f'(a-m)) \end{aligned}$$

These can be simplified as follows.

$$g(\lambda) = \rho - f'(k) = g(u_g) \tag{4}$$

$$u_c = u_g\ [1 + f'(k) + \pi] \tag{5}$$

A solution to the household problem then consists of a set of functions (c_t, g_t, m_t, k_t) that solve the first-order conditions (4) and (5), the budget constraint, and the CIA constraint if it binds, or $u_c = u_g$ if it does not.

(b) A competitive equilibrium is a set of functions $(c_t, g_t, m_t, k_t, p_t, M_t, x_t)$ that solve 4 household equations, $m = M/p$, $\dot{M} = p\,x = g(M)\,M$, and goods market clearing $f(k) = c + g + \dot{k}$.

(c) The steady state consists of scalars (c, g, m, k, π, x) which are determined near-recursively as follows. A constant real money stock requires $\pi = g(M)$. The Euler equation determines the capital stock: $f'(k) = \rho$. The two consumption flows are determined by $f(k) = c + g$ and $u_c/u_g - 1 = \rho + \pi$.

If the CIA constraint does not bind, then the last equation is $u_c = u_g$, so that $\pi = -\rho$ and the nominal interest rate is zero. This can only happen, if the money growth rate is set to $g(M) = -\rho$. If the CIA constraint binds, then $m = c$.

(d) If the CIA constraint does not bind, then it will bind once the money growth rate is increased. So I only consider the effects when the CIA constraint binds. Clearly, money growth does not affect k. The effect on consumption is determined by $f(k) = c + g$ and $U'(c)/V'(g) = 1 + \rho + g(M)$. It is easy to see that higher money growth increases g and reduces c. Since $m = c$, real balances decline as well. Standard substitution is the intuition.