

Cass-Koopmans Model

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The Growth Model in Continuous Time

- We add optimizing households to the Solow model.
- We first study the planner's problem, then the CE.

Planning Problem

Planning Problem

The social planner maximizes

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \quad (1)$$

subject to the resource constraint

$$\dot{k}_t = f(k_t) - (n + \delta)k_t - c_t \quad (2)$$

$$k_0 \text{ given} \quad (3)$$

$$k_t \geq 0 \quad (4)$$

Planning Problem

The current value Hamiltonian is

$$H(c_t, k_t, \mu_t) = u(c_t) + \mu_t[f(k_t) - (n + \delta)k_t - c_t] \quad (5)$$

The state is k and the control is c .

The optimality conditions are

$$\partial H / \partial c = 0 \Rightarrow u'(c_t) = \mu_t \quad (6)$$

$$\begin{aligned} \dot{\mu}_t &= (\rho - n)\mu_t - \partial H_t / \partial k_t \\ &= \mu_t[\rho - n - f'(k_t) + n + \delta] \\ &= \mu_t[\rho + \delta - f'(k_t)] \end{aligned} \quad (7)$$

A solution consists of functions of time

$$c_t, k_t, \mu_t$$

that satisfy:

- 1 The first-order conditions (2)
- 2 The resource constraint
- 3 The boundary conditions k_0 given and the TVC

$$\lim e^{-(\rho-n)t} \mu_t k_t = 0 \quad (8)$$

Planner: Euler Equation

We eliminate the multiplier.
Differentiating the FOC yields

$$\dot{\mu} = u''(c)\dot{c} \quad (9)$$

and therefore

$$\dot{\mu}/\mu = u''(c)\dot{c}/u'(c) \quad (10)$$

Substitute into the law of motion for μ :

$$\dot{c} = u'(c)/u''(c) \cdot [\rho + \delta - f'(k)] \quad (11)$$

$$g(c) = [f'(k) - \delta - \rho] / \sigma \quad (12)$$

where

$$\sigma = -u''_c c / u' \quad (13)$$

$$= -\frac{du'(c)}{dc} \frac{c}{u'(c)} \quad (14)$$

is the intertemporal elasticity of substitution (and the coefficient of relative risk aversion).

Note: $u(c) = c^{1-\phi} / 1 - \phi$ implies $\sigma = \phi$.

$$g(c) = [f'(k) - \delta - \rho] / \sigma \quad (15)$$

Recall the discrete time version:

$$\frac{c_{t+1}}{c_t} = [\beta R]^{1/\sigma} \quad (16)$$

The same idea:

- consumption growth rises with the interest rate
- declines with the discount rate.

Planner: Summary

- The planner's problem solves for functions of time $c(t)$ and $k(t)$.
- These satisfy two differential equations

$$g(c) = \frac{f'(k) - \delta - \rho}{\sigma} \quad (17)$$

$$\dot{k} = f(k) - (n + \delta)k - c \quad (18)$$

and two boundary conditions

$$\lim_{t \rightarrow \infty} \beta^t u'(c(t)) k(t) = 0 \quad k_0 \text{ given}$$

- How can we analyze the dynamics of this system?

Phase Diagram

Phase Diagram

- Phase diagrams can be used to analyze the dynamics of systems of 2 differential equations.
- Consider the example

$$\dot{x} = A - ax + by$$

$$\dot{y} = B + cx - dy$$

- Assume $a, b, c, d > 0$.

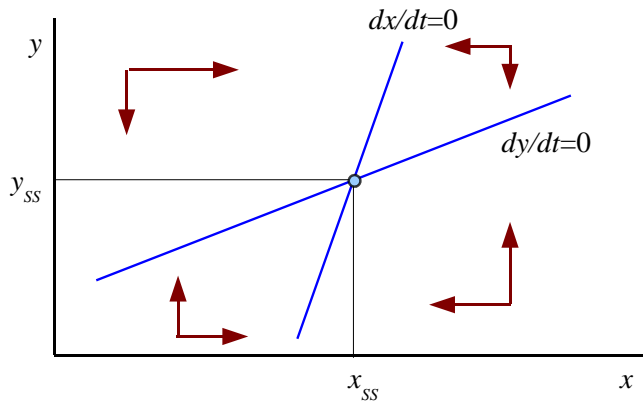
Phase Diagram: Steps

- Step 1: In an (x, y) plane, plot combinations of (x, y) that yield $\dot{x} = 0$ or $\dot{y} = 0$.

$$\dot{x} = 0 \Rightarrow y = \frac{ax - A}{b}$$
$$\dot{y} = 0 \Rightarrow y = \frac{B + cx}{d}$$

- Step 2: Find out in which direction the system moves when off the $\dot{x} = 0$ or $\dot{y} = 0$ lines.
 - raise x : \dot{x} falls - move left
 - raise y : \dot{y} falls - move down
- Step 3: Divide phase diagram into 4 quadrants.
 - draw arrows of movement and think...

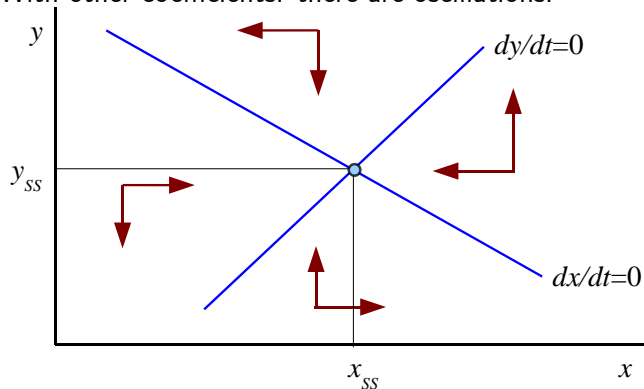
Phase Diagram



The steady state is stable.

Phase Diagram

With other coefficients: there are oscillations.



Phase Diagram: Growth Model

The $\dot{c} = 0$ locus is characterized by

$$f'(k^*) = \rho + \delta \quad (19)$$

The $\dot{k} = 0$ locus is hump-shaped:

$$c = f(k) - (n + \delta)k \quad (20)$$

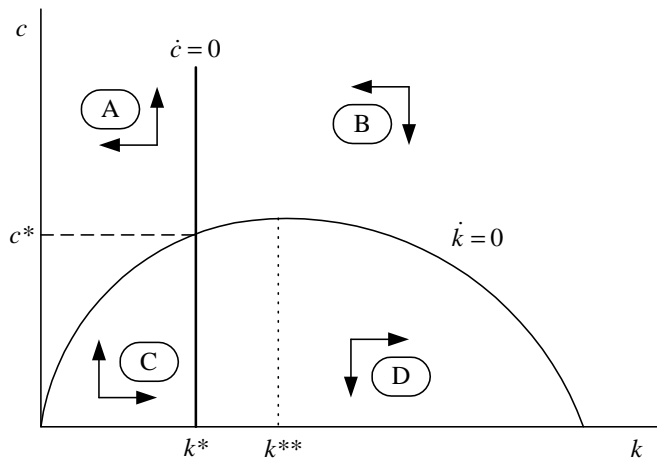
with a maximum at

$$f'(k^*) = n + \delta \quad (21)$$

Since $\rho - n > 0$, the $\dot{c} = 0$ locus lies to the left of the peak of the $\dot{k} = 0$ locus.

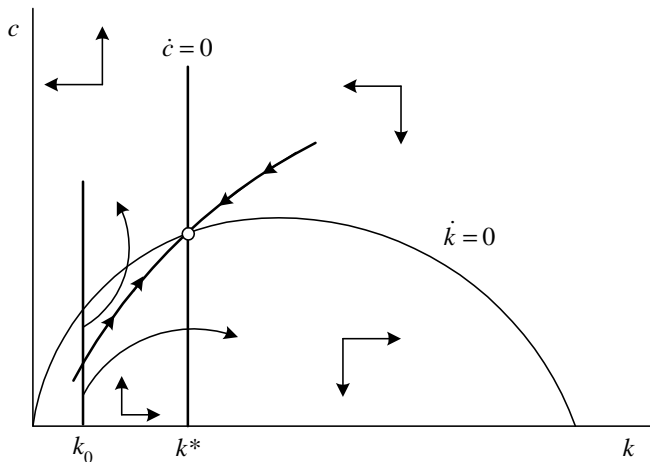
The steady state is located at the intersection of the two curves.

Phase Diagram



- We show that a unique c satisfies the equilibrium conditions for each k .
- Region D can never be reached.
 - c would hit zero in finite time.
- Region A can never be reached.
 - k would hit zero in finite time.
 - Then c would have to jump to 0.

Dynamics



Only one value of c avoids moving into regions A and D for given k .
For this c , the economy converges to the steady state.
Such a system is called "saddle-path stable."

Competitive Equilibrium

Competitive Equilibrium

- Firms solve the same problem as in the Solow model.
- We add a government that imposes lump-sum taxes to finance government spending.
- The budget constraint is $\tau_t = G_t$.

$$\max \int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \quad (22)$$

subject to: k_0 given, the TVC, and the budget constraint

$$\dot{k}_t = w_t + (r_t - \delta - n)k_t - c_t - \tau_t \quad (23)$$

Hamiltonian:

$$H = u(c) + \lambda[w + (r - \delta - n)k - c - \tau] \quad (24)$$

First-order conditions

$$\partial H / \partial c = 0 \Rightarrow u'(c) = \lambda \quad (25)$$

$$\begin{aligned} \dot{\lambda} &= (\rho - n)\lambda - \partial H / \partial k \\ &= \lambda[\rho - n - (r - \delta - n)] \\ &= \lambda(\rho - r + \delta) \end{aligned}$$

Transversality:

$$\lim_{t \rightarrow \infty} e^{-(\rho-n)t} \lambda_t k_t = 0 \quad (26)$$

Eliminate λ :

$$u''(c)\dot{c} = \dot{\lambda} \quad (27)$$

Substitute into the law of motion for λ :

$$\dot{c} = u'(c)/u''(c) \cdot [\rho + \delta - r]$$

or

$$g_c = (r - \delta - \rho)/\sigma \quad (28)$$

Solution: Functions c_t, k_t that solve the Euler equation, the budget constraint, and the boundary conditions.

Competitive Equilibrium

Objects: Functions $c_t, k_t, \tau_t, w_t, r_t$.

Equilibrium conditions:

- Household (2)
- Firm (2)
- Government (1)
- Market clearing (1)

Simplify to obtain two differential equations:

$$\dot{c} = u'(c)/u''(c) \cdot [\rho + \delta - f'(k)] \quad (29)$$

$$\dot{k} = f(k) - (n + \delta)k - c - G \quad (30)$$

The planning solution and the CE coincide (with $G = 0$).

Detrending the Model

Detrending a model

- Consider the Cass Koopmans model with productivity growth:

$$\max \int_0^{\infty} e^{-(\rho-n)t} u(c_t) dt \quad (31)$$

$$\dot{k}_t = F(k_t, A_t) - (n + \delta)k_t - c_t \quad (32)$$

with

$$A_t = e^{gt} \quad (33)$$

- What does the Planner's solution look like?
- The problem: the model has no steady state.
- How can we analyze its dynamics?

Approach 1: Solve and detrend

- Unchanged: the Planner's optimality conditions in terms of original variables:

$$\dot{c}/c = \frac{\frac{\partial F(k,A)}{\partial k} - n - \delta - (\rho - n)}{\sigma(c)} \quad (34)$$

- But we cannot draw the phase diagram without a steady state.
- Solution: detrend the variables to make them stationary.
 - 1 Find the balanced growth rate for each variable.
 - 2 Divide each variable by a scale factor that grows at its balanced growth rate.

Balanced growth rates

- The same as in the Solow model with growth:

$$g(c) = g(k) = g \quad (35)$$

- Define the detrended variables:

$$\tilde{c}_t = c_t/A_t \quad (36)$$

$$\tilde{k}_t = k_t/A_t \quad (37)$$

- Law of motion:

$$\begin{aligned} g(\tilde{k}) &= g(k) - g \\ &= \frac{F(\tilde{k}, 1)A - (n + \delta)\tilde{k}A - \tilde{c}A}{k} - g \\ d\tilde{k}/dt &= f(\tilde{k}) - (n + \delta + g)\tilde{k} - \tilde{c} \end{aligned} \quad (38)$$

- Optimality conditions in terms of detrended variables:

$$\begin{aligned}\frac{d\tilde{c}/dt}{\tilde{c}} &= \frac{\dot{c}}{c} - g \\ &= \frac{f'(\tilde{k}) - \delta - \rho}{\sigma(c)} - g\end{aligned}\quad (39)$$

- This is true because

$$\frac{\partial F(k, A)}{\partial k} = \frac{\partial F(\tilde{k}A, A)}{\partial \tilde{k}} \frac{\partial \tilde{k}}{\partial k} = Af'(\tilde{k}) \frac{1}{A}\quad (40)$$

- Assume CRRA preferences:

$$u(c) = c^{1-\sigma} / (1-\sigma) \quad (41)$$

- Then $\sigma(c) = \sigma$ is constant.
- **CRRA is required for balanced growth** - an important result.
 - Otherwise $\sigma(c)$ is not constant.

Approach 2: Detrend and solve

- Steps:
 - ① Find balanced growth rates - as before.
 - ② Write the economy in detrended variables.
 - ③ Take the first-order conditions.
 - ④ Define the solution.
 - ⑤ Convert back into (undetrended) variables.
- This is useful for solution methods that only work on stationary problems (such as DP).
- Exercise: show that this yields the same answer for the growth model.

Detrending the Model

Summary

In the growth model, optimality conditions change only by adding the 2 occurrences of g :

$$g(\tilde{c}) = \frac{f'(\tilde{k}) - \delta - \rho}{\sigma} - g \quad (42)$$

$$d\tilde{k}/dt = f(\tilde{k}) - (n + \delta + g)\tilde{k} - \tilde{c} \quad (43)$$

Detrending the Model

Why do we care?

- 1 The balanced growth \tilde{k} now depends on preferences:

$$g(\tilde{c}) = 0 \Rightarrow f(\tilde{k}) = \delta + \rho + \sigma g \quad (44)$$

- 2 We see that preferences must be CRRA for a steady state to exist.
- 3 Quantitative differences.

- Acemoglu, "Introduction to Modern Economic Growth," ch. 8. Ch. 8.6 covers the detrended model. Ch. 7 covers Optimal Control.
- Barro & Sala-i-Martin, ch. 2, explains the Cass-Koopmans model in great detail.
- Blanchard & Fischer (1989), ch. 2
- Romer. ch. 2A
- Phase diagram: Barro & Sala-i-Martin ch. 2.6