Two Sector Models

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Two Sector Models

We relax the assumption that there is only one good at each date.

There are no major changes in methods.

Multi-sector models are used to study issues such as:

- technical change that is "embodied" in capital goods,
- human capital,
- international trade.

The Environment

Demographics:

- There is a unit mass of **households** who live forever.

Preferences:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - v_t)$$

- v is work; $1 - v$ is leisure.

Technologies

Consumption goods are produced according to

$$Y_1 = F(K_1, L_1)$$

and capital goods according to

$$Y_2 = G(K_2, L_2)$$

The resource constraints are

$$\begin{aligned} L_{1t} + L_{2t} &= \nu_t \\ K_{1t} + K_{2t} &= K_t \\ Y_{1t} &= c_t \\ Y_{2t} &= K_{t+1} - (1-\delta)K_t \end{aligned}$$

Capital mobility

There is only one type of capital.

It can be moved freely between sectors.

Planning Problem

The **planner** maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - v_t)$$

subject to the resource constraints.

Since capital can be costlessly reallocated between sectors, the state variable is K_t.

The controls $c_t, L_{1t}, L_{2t},$ and φ_t.

φ_t is the fraction of capital employed in sector 1

$$\begin{aligned} K_{1t} &= \varphi_t \, K_t \\ K_{2t} &= (1 - \varphi_t) K_t \end{aligned}$$

Exercise

Suppose that reallocating capital across sectors is not possible.

Solve the planner's problem.

We now need 2 states (K_{1t}, K_{2t}).

Resource constraint:

$$Y_{2t} = K_{1,t+1} + (1-\delta_1)K_{1,t} + K_{2,t+1} + (1-\delta_1)K_{2,t} \tag{1}$$

The difference: $K_{j,t+1}$ has to be chosen in t.

Planning Problem

The Bellman equation is

$$\begin{aligned} V(K) &= \max u(F(\varphi K, L_1), 1 - L_1 - L_2) \\ &\quad + \beta V(K(1-\delta) + G([1-\varphi]K, L_2)) \end{aligned}$$

where the choice variables are L_1, L_2, φ.

Planning Problem

FOCs:

$$\begin{aligned} u_l &= \beta V'(K')G_L = u_c F_L \\ u_c F_K &= \beta V'(K')G_K \end{aligned}$$

Envelope:

$$\begin{aligned} V'(K) &= \varphi F_K u_c + \beta V'(K')\{1-\delta+(1-\varphi)G_K\} \\ &= u_c F_K + (1-\delta)\beta V'(K') \end{aligned}$$

Interpretation...

Planning Problem

Euler equation

$$u_c \frac{F_K}{G_K} = \beta u_c(.') \frac{F_K(.')}{G_K(.')} \{1 - \delta + G_K(.')\}$$

Static conditions

$$F_K / F_L = G_K / G_L$$
$$u_l = u_c F_L$$

Interpretation below...

Solution: Planning Problem

Sequences $\{c_t, v_t, K_{t+1}, \varphi_t, L_{1t}, L_{2t}\}$ that satisfy:

- 3 FOCs;
- 3 feasibility conditions;
- TVC: $\lim_{t\to\infty} \beta^t \; u_c(t) \; K_t = 0$.
- K_0 given.

Intuition: Static condition

$F_K/F_L = G_K/G_L$

The static condition equates marginal rates of substitution in the two sectors.

This is necessary for maximizing output for given inputs.

Intuition: Euler equation

$$u_c \frac{F_K}{G_K} = \beta u_c(.') \frac{F_K(.')}{G_K(.')} \{1 - \delta + G_K(.')\}$$

Consider first the case $F_K = G_K$.

Then we get the conventional Euler equation

$$u_c = \beta \; u_c \left(.'\right) \; \left\{1 - \delta + G_K \left(.'\right)\right\}$$

Intuition: Euler equation - General case

- At any point in time, consumption can be converted into next period capital at a marginal rate of transformation G_K/F_K.
- Period t: Convert 1 unit of c into G_K/F_K units of K'.
- Period $t+1$: Produce an additional

$$\{(1-\delta)+G_K(.')\}G_K/F_K$$

  units of date t+2 capital.
- Convert the additional date t+2 capital into date t+1 consumption at the rate of transformation

$$F_K(.')/G_K(.')$$

- Eat this. This leaves all variables after $t+1$ unchanged.

Planner: Steady State

In steady state, the Euler equation simplifies to

$$\beta\{1-\delta+G_K\}=1$$

Because the MRT, G_K/F_K, is constant this is the same as in the one sector model.

Competitive Equilibrium

Competitive Equilibrium

Watch your units!

Notation

- P_i are the prices of the goods.
- $p_2 = P_2/P_1$.
- RP_1 and wP_1 are the rental prices of capital and labor.

All prices are in units of account.

Competitive Equilibrium

Consumption sector **firms** maximize period profits:

$$\max P_1 \left[Y_1 - RK_1 - wL_1\right]$$

The FOCs are as usual:

$$\begin{aligned} R &= F_K \\ w &= F_L \end{aligned}$$

Competitive Equilibrium

Capital sector firms:

$$\max P_2 Y_2 - P_1 R K_2 - P_1 w L_2$$

Divide through by P_1 to obtain

$$\max p_2 Y_2 - R K_2 - w L_2$$

The FOCs are

$$\begin{aligned} R/p_2 &= G_K \\ w/p_2 &= G_L \end{aligned}$$

Households

The budget constraint is

$$P_{2t}k_{t+1} = P_{2t}(1-\delta)k_t + P_{1t}R_tk_t + P_{1t}(w_tv_t - c_t)$$

Divide through by P_1 to obtain the budget constraint in real terms:

$$p_{2t}k_{t+1} = (1-\delta)p_{2t}k_t + R_tk_t + w_tv_t - c_t$$

Rate of return

At t: Give up p_{2t} units of consumption and buy $dk_{t+1} = 1$

At $t+1$:

- Receive rental income $R_{t+1} dk_{t+1}$.
- Sell the undepreciated capital: $(1-\delta) p_{2,t+1} dk_{t+1}$.

The rate of return is

$$\begin{aligned} 1 + r_{t+1} &= \frac{R_{t+1} + (1-\delta) p_{2,t+1}}{p_{2,t}} \\ &= R_{t+1}/p_{2,t} + (1-\delta)\pi_{t+1} \end{aligned}$$

$\pi_{t+1} \equiv p_{2,t+1}/p_{2,t}$ is the price appreciation of k.

Household problem I

$$V(k) = \max_{k',v} u\left(wv + Rk + (1-\delta)p_2 k - p_2 k', 1-v\right) \tag{2}$$
$$+\beta V\left(k'\right) \tag{3}$$

FOCs:

$$u_1 w = w_2 \tag{4}$$
$$u_1 p_2 = \beta V'\left(k'\right) \tag{5}$$

Envelope:

$$V'(k) = u_1 \times (R + (1-\delta)p_2) \tag{6}$$

Household problem II

Euler:

$$u_1 = \beta u_1 \left(.'\right) \frac{R' + [1-\delta] p_2'}{p_2} \tag{7}$$

$$\beta u_1 \left(.'\right) \left(1 + r'\right) \tag{8}$$

Note that we could have written that down without even deriving it.

- All the household needs to know about an asset is the rate of return.

Market clearing

Labor: $L_{1t} + L_{2t} = \nu_t$.

Capital: $K_{1t} + K_{2t} = K_t$.

Goods:

$$\begin{aligned} Y_{1t} &= c_t \\ Y_{2t} &= K_{t+1} - (1-\delta)K_t \end{aligned}$$

These are just the technological constraints.

Equilibrium Definition

A CE is a sequence of prices

and quantities

which satisfy (11 equations in 10 unknowns):

Equilibrium prices

The firms' FOCs imply that

$$R = p_2 G_K = F_K$$

and therefore

$$p_2 = F_K / G_K$$

In words: the relative price equals the marginal rate of transformation.

Exercise: Show that the solutions of the planning problem and the CE coincide by substituting prices for derivatives of F and G in the planner's FOCs.

One-sector Reduced Form

A One-sector Reduced Form

- We can construct a two sector model that looks very much like a one sector model.
- This requires the assumption

$$G(K, L) = AF(K, L)$$

for some constant A

A One-sector Reduced Form

Then static optimality (same MRS in both sectors)

$$F_K/F_L = G_K/G_L$$

implies

$$k_1 = k_2$$

where $k = K/L$.

The relative price of capital is constant

$$p_2 = 1/A$$

Intuition ...

A One-sector Reduced Form

We can write a single **aggregate resource constraint**:

Define aggregate real output as

$$\begin{aligned} Y &= Y_1 + Y_2/A \\ &= F(K_1, L_1) + F(K_2, L_2) \\ &= (L_1 + L_2) f(k) \\ &= F(K, L) \\ &= c + (K_{t+1} - [1-\delta]K_t)/A \end{aligned}$$

But isn't this adding apples and oranges?

Choose units of capital such that $A = 1$: $\tilde{K} = K/A$.

Then the resource constraint looks like a one sector model:

$$F(\tilde{K}_t, L_t) = c_t + \tilde{K}_{t+1} - (1-\delta)\tilde{K}_t$$

Why is this useful?

We can write down a model with cross-country (or cross-industry) productivity differentials without having to construct a full-blown multi-sector model with endogenous prices.

In the data, the relative **price of capital** varies greatly across countries. We can model that.

Example: Investment specific technical change

Assume that A_t grows at some rate.

Then the relative price of capital $p_{2,t}/1/A_t$ falls over time (as it does in the data).

The model generates an evolution of the industrial structure

- capital intensive sectors grow over time
- e.g. movement from ag to industry.

Greenwood, Hercowitz, and Krusell (1997) find that such technical change accounts for 60 percent of overall productivity growth.

Summary

- Nothing fundamental changes when there are multiple sectors.
- The main additional complexity is in the household budget constraint because there may be capital gains terms.
- The dynamics of two sector models is much more complex than that of one sector models.