

Two Sector Models

Prof. Lutz Hendricks

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Two Sector Models

We relax the assumption that there is only one good at each date.

There are no major changes in methods.

Multi-sector models are used to study issues such as:

- technical change that is “embodied” in capital goods,
- human capital,
- international trade.

- There is a unit mass of **households** who live forever.
- Preferences are

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - v_t)$$

- v is work; $1 - v$ is leisure.

Planning Problem

- Consumption goods are produced according to

$$Y_1 = F(K_1, L_1)$$

- and capital goods according to

$$Y_2 = G(K_2, L_2)$$

- The resource constraints are

$$L_{1t} + L_{2t} = v_t$$

$$K_{1t} + K_{2t} = K_t$$

$$Y_{1t} = c_t$$

$$Y_{2t} = K_{t+1} - (1 - \delta)K_t$$

Planning Problem

- The **planner** maximizes

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - v_t)$$

subject to the resource constraints.

- The planner chooses c_t, L_{1t}, L_{2t} , and φ_t .
- φ_t is the fraction of capital employed in sector 1:

$$\begin{aligned} K_{1t} &= \varphi_t K_t \\ K_{2t} &= (1 - \varphi_t) K_t \end{aligned}$$

- The planner's **state** variable is K_t .
- We would need 2 states (K_{1t}, K_{2t}) if there were a cost of reallocating capital.

The Bellman equation is

$$V(K) = \max u(F(\varphi K, L_1), 1 - L_1 - L_2) \\ + \beta V(K(1 - \delta) + G([1 - \varphi]K, L_2))$$

where the choice variables are L_1 , L_2 , and φ .

FOCs:

$$\begin{aligned}u_l &= \beta V'(K')G_L = u_c F_L \\u_c F_K &= \beta V'(K')G_K\end{aligned}$$

Envelope:

$$V'(K) = \varphi F_K u_c + \beta V'(K')\{1 - \delta + (1 - \varphi)G_K\}$$

Euler equation

$$u_c \frac{F_K}{G_K} = \beta u_c(\cdot) \frac{F_K(\cdot)}{G_K(\cdot)} \{1 - \delta + G_K(\cdot)\}$$

Static condition

$$F_K / F_L = G_K / G_L$$

Solution: Planning Problem

Sequences $\{c_t, v_t, K_{t+1}, \varphi_t, L_{1t}, L_{2t}\}$ that satisfy:

- 2 FOCs;
- 4 feasibility conditions;
- TVC: $\lim_{t \rightarrow \infty} \beta^t u_c(t) K_t = 0$.
- K_0 given.

Intuition: Static condition

- The static condition equates marginal rates of substitution in the two sectors.
- This is necessary for maximizing output for given inputs.

Intuition: Euler equation

$$u_c \frac{F_K}{G_K} = \beta u_c(.) \frac{F_K(.)}{G_K(.)} \{1 - \delta + G_K(.)\}$$

Consider first the case $F_K = G_K$.

- Then we get the conventional Euler equation

$$u_c = \beta u_c(.) \{1 - \delta + G_K(.)\}$$

Intuition: Euler equation - General case

- At any point in time, consumption can be converted into next period capital at a marginal rate of transformation G_K/F_K .
- Period t : Convert 1 unit of c into G_K/F_K units of K' .
- Period $t+1$: Produce an additional

$$\{(1 - \delta) + G_K(\cdot)\} G_K/F_K$$

units of date $t+2$ capital.

- Convert the additional date $t+2$ capital into date $t+1$ consumption at the rate of transformation

$$F_K(\cdot)/G_K(\cdot)$$

- Eat this. This leaves all variables after $t+1$ unchanged.

In steady state, the Euler equation simplifies to

$$\beta\{1 - \delta + G_K\} = 1$$

Because the MRT, G_K/F_K , is constant this is the same as in the one sector model.

Competitive Equilibrium

Notation

- P_i are the prices of the goods.
- $p_2 = P_2/P_1$.
- RP_1 and wP_1 are the rental prices of capital and labor.

Consumption sector **firms** maximize period profits:

$$\max Y_1 - RK_1 - wL_1$$

The FOCs are as usual:

$$R = F_K$$

$$w = F_L$$

Competitive Equilibrium

Capital sector firms:

$$\max P_2 Y_2 - P_1 R K_2 - P_1 w L_2$$

Divide through by P_1 to obtain

$$\max p_2 Y_2 - R K_2 - w L_2$$

The FOCs are

$$R/p_2 = G_K$$

$$w/p_2 = G_L$$

The budget constraint is

$$P_{2t}k_{t+1} = P_{2t}(1 - \delta)k_t + P_{1t}R_tk_t + P_{1t}(w_tv_t - c_t)$$

Divide through by P_1 to obtain the budget constraint in real terms:

$$p_{2t}k_{t+1} = (1 - \delta)p_{2t}k_t + R_tk_t + w_tv_t - c_t$$

Rate of return

At t : Give up some consumption and save it

$$dk_{t+1} = -dc_t/p_{2t}$$

At $t+1$:

- Receive rental income $R_{t+1}dk_{t+1}$.
- The value of undepreciated capital: $(1 - \delta)p_{2,t+1}dk_{t+1}$.
- Additional consumption:

$$dc_{t+1} = -\{(1 - \delta)p_{2,t+1} + R_{t+1}\}/p_{2t} dc_t$$

The rate of return is

$$\begin{aligned} 1 + r_{t+1} &= \frac{dc_{t+1}}{dc_t} \\ &= R_{t+1}/p_{2,t} + (1 - \delta)\pi_{t+1} \end{aligned}$$

$\pi_{t+1} \equiv p_{2,t+1}/p_{2,t}$ is the price appreciation of k .

To solve the problem it helps to write the budget constraint in terms of assets:

$$\begin{aligned} a_{t+1} &= p_{2,t+1} k_{t+1} \\ &= (p_{2,t+1}/p_{2,t}) p_{2,t} k_{t+1} \\ &= \pi_{t+1} \{ (1 - \delta) a_t + R_t / p_{2,t} a_t + w_t v_t - c_t \} \end{aligned}$$

Household Problem

The Lagrangian for this problem is:

$$\sum_{t=0}^{\infty} \beta^t u \left(\begin{array}{l} (1-\delta)a_t + R_t/p_{2,t}a_t \\ +w_tv_t - a_{t+1}/\pi_{t+1}, 1-v_t \end{array} \right)$$

FOCs:

$$\begin{aligned} \beta u_c(t) \{1 - \delta + R_t/p_{2,t}\} &= u_c(t-1)/\pi_t \\ u_l/u_c &= w \end{aligned}$$

The Euler equation can be written as

$$u_c(t) = \beta(1 + r_{t+1})u_c(t+1)$$

- Labor: $L_{1t} + L_{2t} = v_t$.
- Capital: $K_{1t} + K_{2t} = K_t = a_t/p_{2t}$.
- Goods:

$$Y_1 = c$$

$$Y_{2t} = K_{t+1} - (1 - \delta)K_t$$

Equilibrium Definition

A CE is a sequence of prices (w_t, R_t, p_{2t}) and quantities

$$(L_{1t}, L_{2t}, K_{1t}, K_{2t}, K_t, c_t, v_t, a_t)$$

which satisfy (12 equations in 11 unknowns):

- 2 FOCs for each type of firm
- 2 household FOCs and 1 budget constraint
- 4 market clearing conditions
- the identity $K_{1t} + K_{2t} = K_t$.

The firms' FOCs imply that

$$R = p_2 G_K = F_K$$

and therefore

$$p_2 = F_K / G_K$$

In words: the relative price equals the marginal rate of transformation.

Exercise: Show that the solutions of the planning problem and the CE coincide by substituting prices for derivatives of F and G in the planner's FOCs.

A One-sector Reduced Form

- We can construct a two sector model that looks very much like a one sector model.
- This requires the assumption

$$G(K, L) = AF(K, L)$$

for some constant A .

A One-sector Reduced Form

- Then static optimality

$$F_K/F_L = G_K/G_L$$

implies

$$k_1 = k_2$$

where $k = K/L$.

- The relative price of capital is constant

$$p_2 = 1/A$$

A One-sector Reduced Form

- We can write a single **aggregate resource constraint**:
- Define aggregate real output as

$$\begin{aligned} Y &= Y_1 + Y_2/A \\ &= F(K_1, L_1) + F(K_2, L_2) \\ &= (L_1 + L_2)f(k) \\ &= F(K, L) \\ &= c + (K_{t+1} - [1 - \delta]K_t)/A \end{aligned}$$

- Choose units of capital such that $A = 1$: $\tilde{K} = K/A$.
- Then the resource constraint looks like a one sector model:

$$Y_t = c_t + \tilde{K}_{t+1} - (1 - \delta)\tilde{K}_t = F(\tilde{K}_t, L_t)$$

Why is this useful?

We can write down a model with cross-country (or cross-industry) productivity differentials without having to construct a full-blown multi-sector model with endogenous prices.

In the data, the relative **price of capital** varies greatly across countries. We can model that.

We can study **investment specific technical change**.

- Assume that A grows at some rate.
- Then the relative price of capital falls over time (as it does in the data).
- The model generates an evolution of the industrial structure (e.g. movement from ag to industry).
- Greenwood, Hercowitz, and Krusell (1997) find that such technical change accounts for 60 percent of overall productivity growth.

- Nothing fundamental changes when there are multiple sectors.
- The main additional complexity is in the household budget constraint because there may be capital gains terms.
- The dynamics of two sector models is much more complex than that of one sector models.