

Review Questions: Two-Sector Models

Econ720. Fall 2011. Prof. Lutz Hendricks

Question 1. Habit Formation¹

Consider an economy composed of a continuum of infinitely lived, identical households who maximize discounted utility. At date 0 the household's preferences are:

$$\sum_{t=0}^{\infty} \beta^t \frac{[(c_t - x_t)(1 - v_t)^{\rho}]^{1-\sigma}}{1-\sigma},$$

where v_t denotes time allocated to the production of consumption or investment goods and x is the habit stock: $x_t = b c_{t-1}$. Its interpretation is: If the household consumed a lot in the past, it dislikes the idea of reducing consumption today.

There are two production sectors, one produces consumption goods according to $c_t = (\varphi_t K_t)^{\alpha} (\psi_t v_t)^{1-\alpha}$. The other sector produces new capital according to

$$K_{t+1} = (1 - \delta) K_t + ([1 - \varphi_t] K_t)^{\alpha} ([1 - \psi_t] v_t)^{1-\alpha}.$$

- Formulate and solve the planning problem using a Lagrangean. Interpret the Euler equations.
- Formulate and solve the planning problem using Dynamic Programming.
- Formulate the competitive equilibrium. Show that the Euler equations coincide with those for the planning problem.

Answer: Habit Formation

Note that the two technologies are the same, so that a one-sector reduced form exists.

Also note that in a Dynamic Programming setup the state vector is (K, x) , not just K .

- The first trick is to note that this is really a one-sector economy because the technologies in both sectors are identical. We can therefore combine the two resource constraints into

$$K_{t+1} = (1 - \delta) K_t + K_t^{\alpha} v_t^{1-\alpha} - c_t.$$

It also helps to substitute x out of the utility function. The **Lagrangean** is then

$$\begin{aligned} \Gamma = & \sum_{t=0}^{\infty} \beta^t \frac{[(c_t - b c_{t-1})(1 - v_t)^{\rho}]^{1-\sigma}}{1-\sigma} \\ & + \sum_{t=0}^{\infty} \lambda_t \{(1 - \delta) K_t + K_t^{\alpha} v_t^{1-\alpha} - c_t - K_{t+1}\} \end{aligned}$$

The first-order conditions are therefore:

$$c \quad \beta^t u_c(t) - \lambda_t - b \beta^{t+1} u_x(t+1) = 0$$

¹ Based on a 1996 ASU prelim question.

$$v \quad \beta^t u_v(t) = \lambda_t MPL_t$$

$$K \quad \lambda_{t-1} = \lambda_t (MPK_t + 1 - \delta)$$

The only non-standard term is the last one in the FOC for c . The Euler equation is:

$$u_c(t) - b\beta u_c(t+1) = [\beta u_c(t+1) - b\beta^2 u_c(t+2)][MPK_{t+1} + 1 - \delta].$$

Consider the following perturbation (as usual): Consume one unit less at t and $dc_{t+1} = [MPK_{t+1} + 1 - \delta]$ units more at $t+1$. This leaves consumption beyond $t+1$ unchanged. At t the loss of utility is the usual one, $u_c(t)$. At $t+1$, the household gains not only dc_{t+1} in consumption, but he also has higher utility because the habit stock is lower by $b dc_t$. Finally, the habit stock at $t+2$ is higher by $b dc_{t+1}$. Note that there is no effect beyond $t+2$ because the habit stock does not change at those dates.

There will also be a second Euler equation for u_v , but we derive that in the next section.

$$(b) \quad V(K, x) = u(c - x, v) + \beta V((1 - \delta)K + K^\alpha v^{1-\alpha} - c, b c)$$

$$\text{FOC: } u_c = \beta V_K(\cdot) + \beta b V_x(\cdot), \quad u_v = \beta V_K(\cdot) MPL,$$

$$\text{Envelope: } V_K = \beta V_K(\cdot)[(1 - \delta) + MPK], \quad V_x = -u_c + \beta V_x(\cdot)b$$

Use the FOC for V to substitute the V_K out of the first envelope equation:

$$\beta V_K(\cdot) = u_v / MPL = u'_v / MPL' [1 - \delta + MPK']$$

Interpretation: Give up one unit of leisure today and produce an additional MPL units of capital. Next period, this allows to reduce labor input by the additional output produced by the additional K $[1 - \delta + MPK']$ divided by MPL' .

To get the Euler equation for u_c , use the FOC for c to substitute out V_x in the second envelope condition:

$$\begin{aligned} V_x(\cdot) &= -u'_c + u'_c - \beta V_K(\cdot) \\ &= (u_c - \beta V_K(\cdot)) / (b\beta) \end{aligned}$$

$$\Rightarrow \quad u_c = \beta V_K(\cdot) + b\beta^2 V_K(\cdot)$$

(c) To be written.

Question 2. Human Capital

Consider an economy in discrete time with a single representative household. Preferences are given by $\sum_{t=0}^{\infty} \beta^t u(c_t)$. The household has one unit of time each period which is spent on working $(1-v)$ or human capital accumulation (v) . Human capital evolves according to $h_{t+1} = G(v_t \cdot h_t)$, where G obeys Inada conditions. The household also owns physical capital which evolves according to $a_{t+1} = R_t a_t + w_t (1 - v_t) h_t - c_t$.

(a) Characterize optimal household behavior using Dynamic Programming.

(b) A single representative firm produces according to $F(K_t, L_t) + (1 - \delta_k) K_t = c_t + K_{t+1}$, renting capital at price r_t and efficiency units of labor at price w_t . A household supplies h efficiency units of labor per unit of time, i.e., $L_t = (1 - v_t) h_t$. F is constant returns to scale and obeys Inada conditions. Derive the firm's first order conditions.

(c) Define a competitive equilibrium.

(d) Assume $G = B(vh)^\alpha$ with $B > 0$ and $0 < \alpha < 1$. Also assume $F = AK^\theta L^{1-\theta}$ with $A > 0$ and $0 < \theta < 1$. Derive the solution for the steady state $k = K/L$ in closed form. Derive 2 additional equations that could be solved for v and h .

Answer: Human Capital

(a) $V(a, h) = \max u(c) + \beta V(Ra + w(1 - v)h - c, G(vh))$.

FOCs are: $u'(c) = \beta V_a(\cdot)$, $\beta V_a(\cdot) wh = \beta V_h(\cdot) G'(vh)h$.

Envelope conditions are: $V_a = \beta V_a(\cdot) R$, $V_h = \beta V_a(\cdot) w(1 - v) + \beta V_h(\cdot) G'(vh)v$.

The Euler equation is standard: $u'(c) = \beta R' u'(c')$. There is a static condition as well:

$$\begin{aligned} \beta V_h(\cdot) &= \beta V_a(\cdot) w / G'(vh) \\ &= \beta^2 V_a(\cdot) w' (1 - v') + G'(v'h') v' \beta^2 V_a(\cdot) w' / G'(v'h') \\ &= \beta^2 V_a(\cdot) w' \end{aligned}$$

$$\Rightarrow u'(c) w / G'(\cdot) = \beta u'(c') w'$$

$$\Rightarrow w' G'(vh) / w = R',$$

which requires that the rate of return of investing in human and physical capital must be the same (and relies crucially on the assumption that h fully depreciates).

A solution to the household problem is a sequence (c_t, v_t, h_t, a_t) that satisfies the Euler equation, the static condition, and the 2 laws of motion.

(b) This is standard: $r = f'(k)$ and $w = f(k) - f'(k)k$, where $k = K/L$.

(c) A CE is a sequence of quantities (h, a, v, c, K, L) and prices (R, r, w) that satisfy:

- 4 household conditions
- 2 firm conditions
- $R = 1 + r - \delta_k$
- $K = a$; $L = (1 - v)h$
- $F(K_t, L_t) + (1 - \delta_k) K_t = c_t + K_{t+1}$

$$(d) \boxed{f'(k) = Ak^{\theta-1} = 1/\beta - 1 + \delta_k} \Rightarrow k_{ss} = \left(\frac{A}{1/\beta - 1 + \delta_k} \right)^{1/(1-\theta)}.$$

The other 2 equations are: $R = G'(vh) = \alpha(vh)^{\alpha-1} = 1/\beta$ and $h' = h = B(vh)^\alpha$.