

## Review Questions: Two-Sector Models

Econ720. Fall 2009. Prof. Lutz Hendricks

### Question 1. Habit Formation<sup>1</sup>

Consider an economy composed of a continuum of infinitely lived, identical households who maximize discounted utility. At date 0 the household's preferences are:

$$\sum_{t=0}^{\infty} \beta^t \frac{[(c_t - x_t)(1 - v_t)^{\rho}]^{1-\sigma}}{1-\sigma},$$

where  $v_t$  denotes time allocated to the production of consumption or investment goods and  $x$  is the habit stock:  $x_t = b c_{t-1}$ . Its interpretation is: If the household consumed a lot in the past, it dislikes the idea of reducing consumption today.

There are two production sectors, one produces consumption goods according to  $c_t = (\varphi_t K_t)^{\alpha} (\psi_t v_t)^{1-\alpha}$ . The other sector produces new capital according to

$$K_{t+1} = (1 - \delta) K_t + ([1 - \varphi_t] K_t)^{\alpha} ([1 - \psi_t] v_t)^{1-\alpha}.$$

- Formulate and solve the planning problem using a Lagrangean. Interpret the Euler equations.
- Formulate and solve the planning problem using Dynamic Programming.
- Formulate the competitive equilibrium. Show that the Euler equations coincide with those for the planning problem.

### Answer: Habit Formation

Note that the two technologies are the same, so that a one-sector reduced form exists.

Also note that in a Dynamic Programming setup the state vector is  $(K, x)$ , not just  $K$ .

- The first trick is to note that this is really a one-sector economy because the technologies in both sectors are identical. We can therefore combine the two resource constraints into

$$K_{t+1} = (1 - \delta) K_t + K_t^{\alpha} v_t^{1-\alpha} - c_t.$$

It also helps to substitute  $x$  out of the utility function. The **Lagrangean** is then

$$\Gamma = \sum_{t=0}^{\infty} \beta^t \frac{[(c_t - b c_{t-1})(1 - v_t)^{\rho}]^{1-\sigma}}{1-\sigma} + \sum_{t=0}^{\infty} \lambda_t \{(1 - \delta) K_t + K_t^{\alpha} v_t^{1-\alpha} - c_t - K_{t+1}\}$$

The first-order conditions are therefore:

$$c \quad \beta^t u_c(t) - \lambda_t - b \beta^{t+1} u_x(t+1) = 0$$

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<sup>1</sup> Based on a 1996 ASU prelim question.

$$v \quad \beta^t u_v(t) = \lambda_t MPL_t$$

$$K \quad \lambda_{t-1} = \lambda_t (MPK_t + 1 - \delta)$$

The only non-standard term is the last one in the FOC for  $c$ . The Euler equation is:

$$u_c(t) - b\beta u_c(t+1) = [\beta u_c(t+1) - b\beta^2 u_c(t+2)][MPK_{t+1} + 1 - \delta].$$

Consider the following perturbation (as usual): Consume one unit less at  $t$  and  $dc_{t+1} = [MPK_{t+1} + 1 - \delta]$  units more at  $t+1$ . This leaves consumption beyond  $t+1$  unchanged. At  $t$  the loss of utility is the usual one,  $u_c(t)$ . At  $t+1$ , the household gains not only  $dc_{t+1}$  in consumption, but he also has higher utility because the habit stock is lower by  $b dc_t$ . Finally, the habit stock at  $t+2$  is higher by  $b dc_{t+1}$ . Note that there is no effect beyond  $t+2$  because the habit stock does not change at those dates.

There will also be a second Euler equation for  $u_v$ , but we derive that in the next section.

$$(b) V(K, x) = u(c - x, v) + \beta V((1 - \delta)K + K^\alpha v^{1-\alpha} - c, bc)$$

$$\text{FOC: } u_c = \beta V_K(\cdot) + \beta b V_x(\cdot), \quad u_v = \beta V_K(\cdot) MPL,$$

$$\text{Envelope: } V_K = \beta V_K(\cdot)[(1 - \delta) + MPK], \quad V_x = -u_c + \beta V_x(\cdot)b$$

Use the FOC for  $V$  to substitute the  $V_K$  out of the first envelope equation:

$$\beta V_K(\cdot) = u_v / MPL = u'_v / MPL' [1 - \delta + MPK']$$

Interpretation: Give up one unit of leisure today and produce an additional  $MPL$  units of capital. Next period, this allows to reduce labor input by the additional output produced by the additional  $K$   $[1 - \delta + MPK']$  divided by  $MPL'$ .

To get the Euler equation for  $u_c$ , use the FOC for  $c$  to substitute out  $V_x$  in the second envelope condition:

$$\begin{aligned} V_x(\cdot) &= -u'_c + u'_c - \beta V_K(\cdot) \\ &= (u_c - \beta V_K(\cdot)) / (b\beta) \end{aligned}$$

$$\Rightarrow \quad u_c = \beta V_K(\cdot) + b\beta^2 V_K(\cdot)$$

(c) To be written.

## Question 2. Human Capital

Consider an economy in discrete time with a single representative household. Preferences are given by  $\sum_{t=0}^{\infty} \beta^t u(c_t)$ . The household has one unit of time each period which is spent on working  $(1-v)$  or human capital accumulation ( $v$ ). Human capital evolves according to  $h_{t+1} = G(v_t \cdot h_t)$ , where  $G$  obeys Inada conditions. The household also owns physical capital which evolves according to  $a_{t+1} = R_t a_t + w_t (1 - v_t) h_t - c_t$ .

(a) Characterize optimal household behavior using Dynamic Programming.

(b) A single representative firm produces according to  $F(K_t, L_t) + (1 - \delta_k) K_t = c_t + K_{t+1}$ , renting capital at price  $r_t$  and *efficiency* units of labor at price  $w_t$ . A household supplies  $h$  efficiency units of labor per unit of time, i.e.,  $L_t = (1 - v_t) h_t$ .  $F$  is constant returns to scale and obeys Inada conditions. Derive the firm's first order conditions.

(c) Define a competitive equilibrium.

(d) Assume  $G = B(vh)^\alpha$  with  $B > 0$  and  $0 < \alpha < 1$ . Also assume  $F = AK^\theta L^{1-\theta}$  with  $A > 0$  and  $0 < \theta < 1$ . Derive the solution for the steady state  $k = K/L$  in closed form. Derive 2 additional equations that could be solved for  $v$  and  $h$ .

### Answer: Human Capital

(a)  $V(a, h) = \max u(c) + \beta V(Ra + w(1 - v)h - c, G(vh))$ .

FOCs are:  $u'(c) = \beta V_a(\cdot)$ ,  $\beta V_a(\cdot) wh = \beta V_h(\cdot) G'(vh)h$ .

Envelope conditions are:  $V_a = \beta V_a(\cdot) R$ ,  $V_h = \beta V_a(\cdot) w(1 - v) + \beta V_h(\cdot) G'(vh)v$ .

The Euler equation is standard:  $u'(c) = \beta R' u'(c')$ . There is a static condition as well:

$$\begin{aligned} \beta V_h(\cdot) &= \beta V_a(\cdot) w / G'(vh) \\ &= \beta^2 V_a(\cdot) w' (1 - v') + G'(v'h') v' \beta^2 V_a(\cdot) w' / G'(v'h') \\ &= \beta^2 V_a(\cdot) w' \end{aligned}$$

$$\Rightarrow u'(c) w / G'(\cdot) = \beta u'(c') w'$$

$$\Rightarrow w' G'(vh) / w = R',$$

which requires that the rate of return of investing in human and physical capital must be the same (and relies crucially on the assumption that  $h$  fully depreciates).

A solution to the household problem is a sequence  $(c_t, v_t, h_t, a_t)$  that satisfies the Euler equation, the static condition, and the 2 laws of motion.

(b) This is standard:  $r = f'(k)$  and  $w = f(k) - f'(k)k$ , where  $k = K/L$ .

(c) A CE is a sequence of quantities  $(h, a, v, c, K, L)$  and prices  $(R, r, w)$  that satisfy:

- 4 household conditions
- 2 firm conditions
- $R = 1 + r - \delta_k$
- $K = a$ ;  $L = (1 - v)h$
- $F(K_t, L_t) + (1 - \delta_k) K_t = c_t + K_{t+1}$

$$(d) f'(k) = Ak^{\theta-1} = 1/\beta - 1 + \delta_k \Rightarrow k_{ss} = \left( \frac{A}{1/\beta - 1 + \delta_k} \right)^{1/(1-\theta)}.$$

The other 2 equations are:  $R = G'(vh) = \alpha (vh)^{\alpha-1} = 1/\beta$  and  $h' = h = B(vh)^\alpha$ .