Review Questions: Two Sector Models

Econ720. Fall 2020. Prof. Lutz Hendricks

1 A Planning Problem

The economy is populated by a unit mass of infinitely lived households with preferences given by

$$\sum_{t=0}^{\infty} \beta^t u(c_{Mt}, c_{Ht})$$

where c_{jt} denotes consumption of good j. The household has a unit time endowment in each period.

There are two goods in the economy, indexed by $j = M, H$. The production function for good M is $F(k_{Mt}, h_{Mt})$; it is used for investment and consumption (c_{Mt}). The production function for good H is $G(k_{Ht}, h_{Ht})$; it is consumed as c_{Ht}. k_{jt} denotes capital input in sector j and h_{jt} denotes labor input. Capital goods depreciate at the common rate δ.

(a) Assume that capital cannot be moved between sectors. Once installed in sector j it stays there forever. Formulate the Dynamic Programming problem solved by a central planner.

(b) For the remainder of the question assume that capital can be moved freely between sectors. Formulate the planner's Dynamic Program.

(c) Define a solution to the Planner's problem.

1.1 Answer Sketch: Planning Problem

(a) The planner solves (in sequence language):

$$\max \sum_{t=0}^{\infty} \beta^t u(c_{Mt}, c_{Ht})$$

subject to

$$\begin{aligned} c_{Ht} &= G(k_{Ht}, h_{Ht}) \\ k_{jt+1} &= (1-\delta)\, k_{jt} + i_{jt} \\ i_{jt} &\geq 0 \\ c_{Mt} + i_{Mt} + i_{Ht} &= F(k_{Mt}, h_{Mt}) \end{aligned}$$

There are other ways of writing this. The state variables are both capital stocks. The Dynamic Program is therefore:

$$V(k_M, k_H) = \max u\left(F(k_M, h_M) - i_M - i_H,\, G(k_H, h_H)\right) + \beta V\left((1-\delta)\, k_M + i_M, (1-\delta)\, k_H + i_H\right)$$

subject to $i_j \geq 0$.

(b) The constraint set changes if capital can be moved between sectors. Effectively, the non-negativity constraints on investment are dropped. But it is then more convenient to write the constraints as

$$\begin{aligned} c_{Ht} &= G(k_{Ht}, h_{Ht}) \\ k_{t+1} &= (1-\delta)\, k_t + F(k_t - k_{Ht}, 1 - h_{Ht}) - c_{Mt} \end{aligned}$$

The Dynamic Programming problem is now

$$V(k) = \max u\left[(1-\delta)\, k + F(k - k_H, 1 - h_H) - k',\, G(k_H, h_H)\right] + \beta\, V(k')$$

(c) The first order conditions are

$$u_M\, F_k = u_H\, G_K \tag{1}$$

$$u_M\, F_H = u_H\, G_H \tag{2}$$

$$u_M = \beta\, V'(k') \tag{3}$$

The envelope condition is

$$V'(k) = u_M \; [(1-\delta) + F_K]$$

Combining the last 2 equations yields the standard Euler equation

$$u_M = \beta \, u_M(.') \; [(1-\delta) + F_K(.')] \tag{4}$$

A solution to the planner's problem (in sequence language) consists of sequences $\{k_t, k_{Ht}, c_{Mt}, c_{Ht}\}$ which solve the first-order conditions (1) through (4) and the constraint $c_{Ht} = G(k_{Ht}, h_{Ht})$.

2 Consumption Taxes in a Growth Model

Consider the following version of the growth model. There is a single representative agent with preferences given by:

$$\sum_{t=0}^{\infty} \beta^t \log c_t$$

where c_t is consumption in period t, and $0 < \beta < 1$. The worker is endowed with one unit of time in each period but does not value leisure.

There are two production sectors. One sector produces the consumption good using a Cobb-Douglas technology:

$$c_t = k_{ct}^{\theta} n_{ct}^{1-\theta}$$

where k_{ct} and n_{ct} are capital and labor inputs to this sector at time t respectively. The other sector produces capital goods also using a Cobb-Douglas technology:

$$i_t = A k_{it}^{\eta} n_{it}^{1-\eta}$$

where k_{it} and n_{it} are capital and labor inputs to the investment sector. Feasibility requires:

$$\begin{aligned} (1-\delta)k_t + i_t &= k_{t+1} \\ k_{ct} + k_{it} &= k_t \\ n_{ct} + n_{it} &= 1 \end{aligned}$$

where δ is the depreciation rate for physical capital. Thus, we are assuming that capital is completely mobile across sectors. The initial capital stock k_0 is given.

(a) Define a competitive equilibrium for this economy in sequence form.

(b) Define a steady state competitive equilibrium for this economy. Derive an equation to characterize the steady state value of the capital stock.

(c) Assume that the government places a proportional tax on consumption expenditures equal to τ_c and then simply throws away the tax revenues. How will this affect the steady state values for the capital stock, investment and consumption? Justify your answer.

2.1 Answer Sketch: Consumption Tax

(a) The numeraire is capital. The price of consumption is p_t. The household maximizes discounted utility subject to

$$k_{t+1} = R_{t+1} \, k_t + w_t - p_t \, c_t$$

The Euler equation is

$$u'(c_t) = \beta \, R_{t+1} \, u'(c_{t+1}) \, p_t / p_{t+1}$$

Firms in sector j solve

$$\max \; p_j \, F(k_j, n_j) - r \, k_j - w \, n_j$$

First order conditions are

$$
\begin{aligned}
w/p_j &= f(x_j) - f'(x_j)\, x_j \\
r/p_j &= f'(x_j) \\
x_j &= k_j/n_j
\end{aligned}
$$

Competitive Equilibrium: Sequences $\{c_t, k_t, k_{jt}, n_{jt}, R_t, r_t, w_t, p_t\}$ which satisfy:

2 household conditions

4 firm conditions

Market clearing: Labor. $c = k_c^{\theta}\, n_c^{1-\theta}$. $k_{+1} = A\, k_i^{\eta}\, n_i^{1-\eta} + (1-\delta)\, k$.

Identities: $k = k_i + k_c$. $R = 1 + r - \delta$.

(b) Steady state: A steady state consists of the same 10 variables (without the time subscripts), which satisfy the same 11 conditions. The Euler equation becomes $\beta\, R = 1$. The investment firm's FOC determines the capital-labor ratio in that sector:

$$r = R - 1 + \delta = A\, \eta\, x_i^{\eta - 1}$$

The market clearing condition for good i implies:

$$\delta\, k = A\, k_i^{\eta}\, n_i^{1-\eta}$$

The requirement that w/r is the same in both sectors yields

$$x_c\, \frac{1-\theta}{\theta} = x_i\, \frac{1-\eta}{\eta}$$

Together with

$$k = k_i + k_c = n_i\, x_i + (1 - n_i)\, x_c$$

we have an equation solving for n_i:

$$k = A\, n_i\, x_i^{\eta}/\delta = n_i\, x_i + (1 - n_i)\, x_c$$

The solution is

$$n_i = x_i^{1-\eta}\, \delta/A$$

Hence, $k = x_i$.

(c) Consumption tax: The only change is in the household budget constraint, where prices are replaced with $(1+\tau)\, p_t$. This does not affect the Euler equation or any of the other equations used in the derivation of the steady state value of k. The only change is that consumption falls by the amount of the tax.

3 Two technologies

Consider an economy with a large number of infinitely lived identical households with preferences given by

$$\sum_{t=0}^{\infty} \beta^t \log c_t.$$

Each household is endowed with k_0 units of capital in period 0 and 1 unit of labor each period. The number of households in period t is N_t, where $N_{t+1} = \eta N_t, \eta > 1$. For simplicity, assume that $N_0 = 1$.

We will consider two alternative technologies for this economy:

Technology 1:

$$Y_t = \gamma^t K_t^{\theta} N_t^{1-\theta}$$

Technology 2:

$$Y_t = \gamma^t K_t^{\mu} N_t^{\phi} L_t^{1-\mu-\phi}$$

In these technologies, $\gamma > 1$ is the rate of exogenous total factor productivity growth, K_t is total (*not* per capita) capital, Y_t is total output, and L_t is the total stock of land. Land is assumed to be a fixed factor; it can not be produced and does not depreciate. To simplify without loss of generality, assume that $L_t = 1$ for all t.

The resource constraint, assuming 100% depreciation of capital each period is given by

$$N_t c_t + K_{t+1} \leq Y_t;$$

with $K_0 = k_0$ given.

1. Suppose that the only technology available is the first one.

   (a) Formulate, as a dynamic programming problem, the social planner's problem that weights all individuals utility equally. That is, the planner weights utility in period t by the number of identical agents alive in that period.

   (b) Characterize the balanced growth path of this economy. ("Characterize" means that you must derive a set of equations that determines all endogenous variables along this path. You do not need to solve these equations.) Solve explicitly for the growth rate of per capital consumption (c_t) along this path.

2. Repeat part 1 using the second technology in place of the first.

3. Compare how the rate of population growth η affects the rate of per capita growth in the two cases. Provide an explanation for your findings.

3.1 Answer: Two technologies[1]

1a. The sequence problem of the social planner is

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} N_t \beta^t \log c_t,$$

subject to

$$N_t c_t + K_{t+1} = \gamma^t K_t^{\theta} N_t^{1-\theta}, \ K_0 \text{ given.}$$

and

$$N_{t+1} = \eta N_t = \eta \left(\eta^t N_0\right) = \eta^{t+1}.$$

The planner's problem can be rewritten as

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} (\beta\eta)^t \log c_t,$$

subject to

$$c_t = \gamma^t \left(\frac{K_t}{\eta^t}\right)^{\theta} - \frac{K_{t+1}}{\eta^t}; \ K_0 \text{ given.} \tag{5}$$

The dynamic program of the social planner is then

$$\begin{aligned} V(K,t) &= \max_{K'} \left\{ \log \left[\gamma^t \left(\frac{K}{\eta^t}\right)^{\theta} - \frac{K'}{\eta^t} \right] + \beta\eta \, V(K', t\prime) \right\}, \\ \text{subject to } t' &= t+1. \end{aligned}$$

1b. To characterize the balanced growth path of this economy:

[1]Due to Joydeep Bhattacharya.

$$\begin{aligned}
\text{Foc w.r.t. } K' &: \frac{1}{\eta^t c} = \beta\eta V_1(K', t'), \\
\text{EC w.r.t } K &: V_1(K,t) = \frac{1}{c}\theta \left(\frac{\gamma}{\eta}\right)^t \left(\frac{K}{\eta^t}\right)^{\theta-1}.
\end{aligned}$$

Combining the two obtains the Euler equation:

$$\frac{1}{\eta^t c} = \frac{\beta}{c'}\theta \left(\frac{\gamma}{\eta}\right)^{t'} \left(\frac{K'}{\eta^{t'}}\right)^{\theta-1},$$

or (reverting to the time notation)

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}}\theta\ \gamma^{t+1} \left(\frac{K_{t+1}}{\eta^{t+1}}\right)^{\theta-1}. \tag{6}$$

Along a balanced growth path (per capita variables grow at a constant rate, say g – remember that K_t is total capital stock):

$$\begin{aligned}
c_t &= g^t \bar{c}; & (7)\\
K_t &= (g\eta)^t \bar{K} & (8)
\end{aligned}$$

This, with (6), implies:

$$1 = \frac{\beta}{g}\theta\gamma^{t+1}\left(g^{t+1}\bar{K}\right)^{\theta-1}. \tag{9}$$

The resource constraint (5) on the balanced growth path

$$\begin{aligned}
g^t\bar{c} &= \gamma^t \left(\frac{(g\eta)^t \bar{K}}{\eta^t}\right)^{\theta} - \frac{(g\eta)^{t+1}\bar{K}}{\eta^t} \Rightarrow & (10)\\
\left(\gamma g^{\theta}\right)^t \bar{K}^{\theta} &= g^t\left(\bar{c} + g\eta\bar{K}\right) & (11)
\end{aligned}$$

Equations (7) - (11) characterize the balanced growth path of this economy.
Observe that (11) can hold for all t is iff $\bar{K}^{\theta} = \bar{c} + g\eta\bar{K}$ and

$$\gamma g^{\theta} = g \Leftrightarrow g = (\gamma)^{\frac{1}{1-\theta}}$$

We could have arrived at the same result by using (9), which will hold for all t iff

$$g = (\gamma)^{\frac{1}{1-\theta}} \text{ and } \bar{K} = \left(\frac{\beta\theta}{g}\right)^{\frac{1}{1-\theta}}$$

2a. The sequence problem of the social planner is

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} N_t\beta^t \log c_t,$$

subject to

$$N_t c_t + K_{t+1} = \gamma^t K_t^{\mu} N_t^{\phi},\ K_0 \text{ given.}$$

and

$$N_{t+1} = \eta N_t = \eta\left(\eta^t N_0\right) = \eta^{t+1}.$$

The planner's problem can be rewritten as

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} (\beta\eta)^t \log c_t,$$

subject to

$$c_t = \gamma^t K_t^{\mu} \left(\eta^t\right)^{\phi-1} - \frac{K_{t+1}}{\eta^t};\ K_0 \text{ given.} \tag{12}$$

The dynamic program of the social planner is then

$$\begin{aligned} V(K,t) &= \max_{K'} \left\{ \log\left[\gamma^t K^{\mu} \left(\eta^t\right)^{\phi-1} - \frac{K'}{\eta^t}\right] + \beta\eta\, V(K',t') \right\}, \\ \text{subject to } t' &= t+1. \end{aligned}$$

2b. To characterize the balanced growth path of this economy:

$$\begin{aligned} \text{Foc w.r.t. } K' &: \frac{1}{\eta^t c} = \beta\eta V_1(K',t'), \\ \text{EC w.r.t } K &: V_1(K,t) = \frac{1}{c}\mu\, \gamma^t K^{\mu-1} \left(\eta^t\right)^{\phi-1}. \end{aligned}$$

Combining the two obtains the Euler equation:

$$\frac{1}{\eta^t c} = \frac{\beta\eta}{c'}\mu\, \gamma^{t'} \left(K'\right)^{\mu-1} \left(\eta^{t'}\right)^{\phi-1},$$

or (reverting to the time notation)

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}}\mu\, \gamma^{t+1} \left(K_{t+1}\right)^{\mu-1} \left(\eta^{t+1}\right)^{\phi}. \tag{13}$$

Once again, on the balanced growth path:

$$c_t = g^t \bar{c}; \tag{14}$$

$$K_t = (g\eta)^t \bar{K}. \tag{15}$$

This, with (13), gets

$$1 = \frac{\beta}{g}\mu\gamma^{t+1} \left((g\eta)^{t+1} \bar{K}\right)^{\mu-1} \left(\eta^{t+1}\right)^{\phi}. \tag{16}$$

The resource constraint (5) on the balanced growth path:

$$g^t \bar{c} = \gamma^t \left((g\eta)^t \bar{K}\right)^{\mu} \left(\eta^t\right)^{\phi-1} - \frac{(g\eta)^{t+1}\bar{K}}{\eta^t} \Rightarrow \tag{17}$$

$$\left(\gamma g^{\mu}\eta^{\phi+\mu-1}\right)^t \bar{K}^{\mu} = g^t\left(\bar{c} + g\eta\bar{K}\right). \tag{18}$$

Equations (7) - (11) characterize the balanced growth path of this economy.
Observe that (11) can hold for all t iff

$$\bar{K}^{\mu} = \bar{c} + g\eta\bar{K} \text{ and} \tag{19}$$

$$g = \gamma g^{\mu}\eta^{\phi+\mu-1} \Leftrightarrow g = \left(\frac{\gamma}{\eta^{1-\phi-\mu}}\right)^{\frac{1}{1-\mu}} \tag{20}$$

Once again, one can arrive at the same result by using (16), which will hold only if g is as above and

$$\bar{K} = \left(\frac{\beta\mu}{g}\right)^{\frac{1}{1-\mu}}$$

3. In the first case η does not affect g. In the second, g is inversely proportional to η: a higher population growth rate reduces the growth rate of per capita variables in the economy (it is even possible that $g < 1$). With the first technology, the economy accumulates capital on a balanced growth path consistent with the growth of *enhanced* labor. One can think of the productivity growth as labor-enhancing (i.e., labor efficiency growing at the rate of

$\gamma^{\frac{1}{1-\theta}}$) and accordingly the capital accumulation takes both population growth and labor productivity growth into account (and grows at the rate of $\eta\,\gamma^{\frac{1}{1-\theta}}$). As a result, per capita output grows at the rate $\gamma^{\frac{1}{1-\theta}}$.

With the second technology, the third factor, land, is fixed. As before $\gamma^{\frac{1}{1-\mu}}$ can be accounted for both labor- and land-enhancing productivity growth. Here, the population as before grows at the rate η and the aggregate capital stock can be made to grow enough to provide for the growing population (i.e., grow at ηg), but the land is fixed. The growth rate of capital then must be adjusted by a factor of population, so that per capita output also grows at g. This is achieved by (20).