The Growth Model: Discrete Time

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The standard growth model

- The neoclassical growth model, aka the standard growth model, is the most important model in macro.
- It underlies entire branches of the literature (parts of growth theory and business cycle theory, for example).
- Here, we study this model in discrete time.
- The **main issues** of this section are:
  - Tools: Dynamic programming
  - The neoclassical growth model

Model structure

There are many versions of the growth model. This is a basic version.

1. Households are identical and live forever.
2. Firms produce a single good using capital and labor.
3. All agents are price takers.
4. All prices are perfectly flexible. All markets clear at all times.

Infinite horizons

- So far we have assumed that agents are finitely lived.
- Analytically more convenient: infinite lifetimes.
- How to justify this?
  - Reduced form of an OLG model with **altruism**.
  - Stochastic deaths (perpetual youth models).

The representative household

Demographics

- There is a continuum of households (uncountably infinite number).
- All households are identical.
    - This is stronger than needed (see notes on aggregation later on).
- We can think of a single, price-taking household.
- The measure of households is 1.
    - Therefore, per capita and aggregate variables are the same.

Preferences

- The household values discounted utility from consumption:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1 \tag{1}$$

- Utility is time separable (for tractability).
- Discounting is exponential (to avoid time consistency problems).
- Time consistency means: If $\{c_t\}_{t=0}^{\infty}$ solves the problem with start date 0, then $\{c_t\}_{t=\tau}^{\infty}$ solves the problem with start date τ.
  - The household does not want to change past plans.

Technology

- The household enters the world with k_0 units of "the good."
- Resource constraint:

$$k_{t+1} = f(k_t) - c_t \tag{2}$$

- We assume Inada conditions for f.
- Capital cannot be negative: $k_t \geq 0$.

Planning Problem

- The planner maximizes discounted utility of the representative household

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad 0 < \beta < 1$$

- Constraints:

$$\begin{aligned} k_{t+1} &= f(k_t) - c_t \\ k_{t+1} &\geq 0 \\ & k_0 \text{ given} \end{aligned}$$

Lagrangian

$$\begin{aligned}\Gamma &= \sum_{t=0}^{\infty} \beta^t u(c_t) \\ &\quad + \sum_{t=0}^{\infty} \lambda_t [f(k_t) - c_t - k_{t+1}]\end{aligned}$$

FOCs for an interior solution:

$$\begin{aligned}\beta^t u'(c_t) &= \lambda_t \\ \lambda_{t+1} f'(k_{t+1}) &= \lambda_t\end{aligned}$$

Euler equation

$$\beta u'(c_{t+1})f'(k_{t+1}) = u'(c_t) \tag{3}$$

- This is exactly the same Euler equation we saw many times before.
- The Euler equation implicitly defines a law of motion for the capital stock:

$$\beta u'(f(k_{t+1}) - k_{t+2})f'(k_{t+1}) = u'(f(k_t) - k_{t+1}) \tag{4}$$

- This is a second order difference equation.

Planner: Solution

- A solution is a sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$.
- These satisfy:
  1. Euler equation
  2. Resource constraint
- We have two difference equations - we need two **boundary conditions**:
  1. k_0 given
  2. **Transversality**:

$$\lim_{t\to\infty} \beta^t \ u'(c_t) \ k_{t+1} = 0 \tag{5}$$

Digression: Transversality Conditions

Consider the following example:

$$\begin{aligned} & \max \sum_{t=0}^{T} \beta^t u(c_t) \\ s.t. \quad k_{t+1} &= e_t + (1 + r_t) k_t - c_t \end{aligned}$$

Digression: Transversality Conditions

Lagrangian

$$\begin{aligned}\Gamma &= \sum_{t=0}^{T} \beta^t u(c_t) \\ &\quad + \sum_{t=0}^{T} \lambda_t \{e_t + (1+r_t) k_t - c_t - k_{t+1}\}\end{aligned}$$

FOCs (necessary):

$$u'(c_t) = \beta \; u'(c_{t+1}) \; (1+r_{t+1})$$

Digression: Transversality Conditions

Solution:

Sequences $\{c_t, k_{t+1}\}$ that satisfy:

- Euler equation
- budget constraint
- k_0 given

Problem 1:

- We allowed the household to choose $c_t \to \infty$ and $k_{t+1} \to -\infty$.
- The household problem has no solution.

Problem 2:

- We have 2 difference equations, but only one boundary condition.
- The solution is not uniquely determined by those.

We need one more boundary condition to ensure that utility is finite.

Digression: Transversality Conditions

Where to find a boundary condition?

- The economics of the problem must suggest the right condition.
- It needs to be imposed as part of the original problem with some economic justification.
- A natural candidate in this example: $k_{T+1} = 0$.
    - The household cannot die in debt.

Infinite horizon case

- What if $T \to \infty$?
- We could impose $\lim_{T \to \infty} k_{T+1} = 0$, but it does not make economic sense.
  - This would prevent the household from perpetually growing its capital stock.
- We need to find a weak condition that makes utility finite.

Infinite horizon case

One solution:

- Write the present value budget constraint as

$$\sum_{t=0}^{T} \frac{c_t}{R_t} = \sum_{t=0}^{T} \frac{e_t}{R_t} + k_0 - \frac{k_{T+1}}{R_{T+1}}$$

  where $R_t = (1+r_1) \times ... \times (1+r_t)$ is a cumulative discount factor.
- Require that $\lim_{T\to\infty} k_{T+1}/R_{T+1} = 0$.
- That ensures finite consumption and picks out a unique solution.

Infinite horizon case

An equivalent solution:

Impose

$$\lim_{T\to\infty} \beta^T \; u'(c_T) \; k_{T+1} = 0$$

This is the same because, by the Euler equation:

$$\beta^T u'(c_T) \; R_T = u'(c_0)$$

Digression: Transversality Conditions

The general point:

- Each dynamic optimization problem requires as many boundary conditions as there are difference equations.
- In economic problems, we are usually short one boundary condition (only k_0 is given).
- We need to find a second boundary condition that is economically justifiable and keeps utility finite.

Dynamic Programming: An Informal Introduction

- The basic idea of DP is to transform a many period optimization problem into a static problem.
- To do so, we summarize the entire future by a **value function**.
- The value function tells us the maximum utility obtainable from tomorrow onwards for any value of the state variables.

Dynamic Programming: An Informal Introduction

- Suppose we solve the planner's problem with starting date t^*:

$$V(k_{t^*}) = \max \sum_{t=t^*}^{\infty} \beta^{t-t^*} u(c_t) + \sum_{t=t^*}^{\infty} \lambda_t [f(k_t) - c_t - k_{t+1}]$$

- The result is an optimal sequence of choice variables (c_t, k_t) and a value function $V(k_{t*})$.
- Given the initial condition k_{t^*} the maximum utility obtainable is $V(k_{t*})$.
- Note that the value function is only a function of the initial capital stock.
- Therefore, k_t is the **state variable** of the problem.

Time consistency

- What if we start the problem at $t^* + 1$?
- Would the planner want to change his optimal choices of k_{t^*+2}, k_{t^*+3}, and so on?
- The answer is obviously "no," ... although I won't prove this just yet.
- A problem with this property is known as **time consistent**:
  - Give the decision maker a choice to change his mind at a later date and he will choose the same actions again.
- Not all optimization problems have this property.
  - For example, changing the specification of discounting easily destroys time consistency (self-control problems arise).

Stationarity

- Compare the value functions obtained from the problems starting at t^* and at $t^* + 1$.
- It is obvious that the function $V(k_{t*})$ does not depend on t^*.
- That is, solving the problem yields the same value function regardless of the starting date.
- Such a problem is called **stationary**.
- Not all optimization problems have this property.
    - For example, if the world ended at some finite date, then the problem at t*+1 looks different from the problem at t*.

Recursive structure

- Now comes the key insight: The right hand side of the Lagrangian can be broken into two terms:

$$\begin{aligned} V(k_{t*}) &= \max u(c_{t*}) + \lambda_{t*}[f(k_{t*}) - c_{t*} - k_{t*+1}] \\ &\quad + \beta V(k_{t*+1}) \end{aligned}$$

- We have
  - one term reflecting current period utility
  - a second term summarizing everything that happens in the future, given optimal behavior, as a function of k_{t^*+1}.
- But since this equation holds for any arbitrary start date, we may drop date subscripts.

Recursive structure

- This yields a **Bellman equation**:

$$\begin{aligned} V(k) &= \max u(c) + \lambda[f(k) - c - k'] \\ &\quad + \beta V(k') \end{aligned}$$

  where the primes denote values in the next period.
- Once we substitute the constraint into the second value function we have

$$V(k) = \max u(c) + \beta V(f(k) - c)$$

- Claim: **Solving the DP is equivalent to solving the original problem** (the Lagrangian).
  - This is proved in various textbooks.

Recursive structure

- The convenient part of this is: we have transformed a multiperiod optimization problem into a two period (almost static) one.
- If we knew the value function, solving this problem would be trivial.
- The bad news is that we have transformed an algebraic equation into a **functional equation**.
- The solution of the problem is a value function V and an optimal policy function

$$c = \phi(k)$$

- Note that c cannot depend on anything other than k, in particular not on k's at other dates, because these don't appear in the Bellman equation.

Solution

A solution to the planner's problem is now a pair of functions

$$[V(k), \phi(k)]$$

that solve the Bellman equation in the following sense.

1. Given $V(k)$, setting $c = \phi(k)$ solves the max part of the Bellman equation.
2. Given that $c = \phi(k)$, the value function solves

$$V(k) = u(\phi(k)) + \beta V(f(k) - \phi(k))$$

The second property says that V is a **fixed point** of the Bellman equation.

Solution

- Think of the Bellman equation as a mapping in a function space:

$$V^{n+1} = T(V^n) = \max u(c) + \beta V^n(f(k) - c)$$

- Given an input argument V^n the mapping produces an output arguments V^{n+1}.
- The solution to the Bellman equation is the V that satisfies $V = T(V)$.

The Planner's Problem with DP

- The Planner's Bellman equation is

$$V(k) = \max u(c) + \beta V(f(k) - c)$$

with state k and control c.

- The FOC for c is

$$u'(c) = \beta \; V'(k')$$

- Problem: we do not know V'.

The Planner's Problem with DP

- Differentiate the Bellman equation to obtain the envelope condition (aka Benveniste-Scheinkman equation):

$$V'(k) = \beta V'(k')f'(k)$$

- Now we can use the FOC to substitute out V' twice:

$$u'(c) = \beta f'(k')u'(c')$$

- We obtain the same Euler equation as from the Lagrangian approach.
- DP also tells us that the optimal c is a function only of k.
- Therefore k' also depends only on k:

$$\begin{aligned} k' &= f(k) - \phi(k) \\ &= h(k) \end{aligned}$$

Capital as control variable

- There are other ways of setting up the Bellman equation.
- With capital as the control:

$$V(k) = \max_{k'} u(f(k) - k') + \beta V(k')$$

- FOC:

$$u'(c) = \beta V'(k')$$

- Envelope condition

$$V'(k) = u'(c) f'(k)$$

- The general point: We cannot choose the state variables, but we can choose the control variables.

Characterizing the Planner's Solution

- It is here where DP has serious advantages over the Lagrangean: one can use results from functional analysis to establish properties of the value function and the policy function.
- In our example it can be shown that the economy converges monotonically from any k_0 to the steady state [Sargent DMT p. 25, fn. 2]:
- Note the difference relative to the OLG economy where much stronger assumptions are needed for this result.

Example: Non-separable Utility

- Consider the following growth economy, modified to include **habit persistence** in consumption.
- The social planner solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, c_{t-1})$$

subject to the feasibility constraints

$$\begin{aligned} c_t + x_t &= f(k_t) \\ k_{t+1} &= x_t + (1-\delta)k_t \end{aligned}$$

- f satisfies Inada conditions.
- Compute and interpret the first-order necessary conditions for the planner's problem.

Example: Non-separable Utility

- This problem does not fit the DP approach without some modification.
- We first solve it using a Lagrangian:

$$
\begin{aligned}
\Gamma &= \textstyle\sum_{t=1}^{\infty} \beta^t u(f(k_t) - x_t, f(k_{t-1}) - x_{t-1}) \\
&\quad + \textstyle\sum_{t=1}^{\infty} \lambda_t (x_t + (1-\delta)k_t - k_{t+1})
\end{aligned}
$$

- First order conditions:

$$
\begin{aligned}
&\beta^t u_1(t, t-1) + \beta^{t+1} u_2(t+1, t) = \lambda_t \\
&f'(k_t)\left(\beta^t u_1(t, t-1) + \beta^{t+1} u_2(t+1, t)\right) = \lambda_t (1-\delta) - \lambda_{t-1}
\end{aligned}
\tag{6}
$$

Example: Non-separable Utility

- Euler equation:

$$\lambda_{t-1} = \lambda_t \left[1 - \delta + f'(k_t)\right]$$

- Define the total marginal utility of consumption as

$$U'(c_{t-1}) = \beta^{t-1} u_1(t-1, t-2) + \beta^t u_2(t, t-1)$$

- The Euler Equation then becomes:

$$U'(c_{t-1}) = U'(c_t)\left(f'(k_t) + 1 - \delta\right) \tag{7}$$

Interpretation

$$U'(c_{t-1}) = U'(c_t)\left(f'(k_t) + 1 - \delta\right) \tag{8}$$

- Give up one unit of c_{t-1}. This costs $U'(c_{t-1})$.
- We can increase x_{t-1} by 1 and raise k_t by 1.
- We eat the results next period at marginal utility $U'(c_t)$.
- We can eat
  - the additional output $f'(k_t)$;
  - the undepreciated capital $1 - \delta$;

Example: Non-separable Utility

A **solution** of the hh problem is:

Sequences $\{x_t, k_t\}$ that satisfy

1. the EE
2. the flow budget constraint.
3. The boundary conditions k_1 given and a TVC:

$$\lim_{t\to\infty} U'(c_t)k_t = 0$$

DP solution

- For DP to work, it must be possible to write the problem as

$$V(s) = \max\ u(s,c) + \beta\ V(s')$$

where s is the state and c is the control.

- The current problem does not fit that pattern:

$$V(k) = \max\ u(c,c_{-1}) + \beta\ V(k')$$

subject to the law of motion

$$\begin{aligned} k' &= f(k) + (1-\delta)k - c \\ x &= f(k) - c \end{aligned}$$

- Nonseparable utility is the problem.

Example: Non-separable Utility

Adding a state variable

The solution is to define an additional state variable

$$z = c_{-1}$$

Then the Bellman equation is

$$\begin{aligned} V(k,z) &= \max_x u(f(k)-x,z) \\ &\quad +\beta V(x+(1-\delta)k, f(k)-x) \end{aligned}$$

FOC

$$u_1(c,z) = \beta V_k(k',z') - \beta V_z(k',z')$$

Adding a state variable

The envelope conditions are

$$
\begin{aligned}
V_z &= u_2(c,z) \\
V_k &= u_1(c,z)f'(k) + \beta V_k(.')(1-\delta) \\
&\quad + \beta V_z(.')f'(k)
\end{aligned}
$$

Now define

$$U'(c) = u_1(c,z) + \beta u_2(c',z')$$

Then substitute out the V_z terms:

$$
\begin{aligned}
U'(c) &= \beta V_k(.') \\
V_k &= U'(c)f'(k) + (1-\delta)\beta V_k(.')
\end{aligned}
$$

Substitute out the V_k terms and we get the same EE as with the Lagrangean.

Guess and Verify

- In very special cases it is possible to solve for the value function in closed form.
- A common case is
  - log utility, $u(c) = \ln(c)$, and
  - Cobb-Douglas technology with full depreciation: $f(k) = Ak^{\theta}$.
- Then we can use the "guess and verify" method.

Guess and Verify

The general approach is:

1. Guess a functional form for V. Stick this into the right-hand-side of the Bellman equation.
2. Solve the max problem given the guess for V. The result is on the left hand side a new value function, V^1.
3. If $V = V^1$ the guess was correct.

Guess and Verify: Example

- Consider the growth model with log utility and Cobb-Douglas production / full depreciation.
- The planner solves:

$$\max \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

$$s.t. \ k_{t+1} = A\, k_t^{\theta} - c_t$$

Guess and Verify: Example

- Guess

$$V(k) = E + F\ln(k)$$

- This is inspired by the hope that V should inherit the form of u.
- Having capital stock k amounts to having output Ak^{θ}, which would suggest

$$\begin{aligned} V(k) &\cong \ln(Ak^{\theta}) \\ &= \ln(A) + \theta\ln(k) \end{aligned}$$

- Note that the guess for V contains some unknown constants (E, F) which we determine as we go along.

Guess and Verify: Example

FOC:

$$u'(c) = \beta V'(k')$$

Envelope condition

$$V'(k) = u'(c)f'(k)$$

Guess and Verify: Example

We can use the FOC to obtain the policy function in terms of the unknown parameters.

$$
\begin{aligned}
u'(f(k) - h(k)) &= \beta V'(h(k)) \\
&\Rightarrow \\
[f(k) - h(k)]^{-1} &= \beta F / h(k) \\
h(k) &= \beta F [f(k) - h(k)]
\end{aligned}
$$

The policy function is

$$
h(k) = \frac{\beta F}{1 + \beta F} A k^{\theta} \tag{9}
$$

The envelope condition then implies

$$
\begin{aligned}
V'(k) &= u'(f(k)-h(k))f'(k) \\
&= \frac{A\theta k^{\theta-1}}{Ak^{\theta}-\beta F/(1+\beta F)Ak^{\theta}} \\
&= \frac{\theta k^{-1}}{1/(1+\beta F)} \\
&= \frac{\theta(1+\beta F)}{k}
\end{aligned}
$$

Guess and Verify: Example

The guess implies $V'(k) = F/k$.

Substituting this into the previous equation yields an expression that can be solved for F:

$$\begin{aligned} F &= \theta(1+\beta F) \\ &= \theta/(1-\theta\beta) \end{aligned}$$

A bit of algebra shows that the policy function becomes

$$h(k) = \theta\beta A k^{\theta}$$

so that consumption is

$$f(k) - h(k) = (1-\theta\beta) A k^{\theta}$$

Summary: Guess and Verify

- Use the guess for V in the FOC to get a policy function that depends on the unknown F: $k' = h(k; F)$.
- Use the guess for V in the envelope condition to get $V'(k; F)$ as a function of the unknown F.
- Get another expression for $V'(k; F)$ by differentiating the guess.
- Use the two expressions for V' to solve for F.

Summary: Guess and Verify

The claim is now that our guess satisfies the Bellman equation with this particular F.

We can verify this directly.

$$
\begin{aligned}
T(V) &= u(f(k)-h(k))+\beta\left\{E+F\ln(f(k)-h(k))\right\} \\
&= \ln\left([1-\theta\beta]Ak^{\theta}\right)+\beta E+\beta\frac{\theta}{1-\theta\beta}\ln\left([1-\theta\beta]Ak^{\theta}\right) \\
&= C_1+\left(\theta+\theta\frac{\theta\beta}{1-\theta\beta}\right)\ln(k) \\
&= C_1+F\ln(k)
\end{aligned}
$$

where C_1 is some constant (which could be used to find E).

In general, it is not necessary to go through this verification step. The fact that a constant F is found is sufficient.

DP vs Lagrangian

What does DP buy us compared with a Lagrangian?

- With **uncertainty**, DP tends to be more convenient than a Lagrangian.
- Results from functional analysis can often be used to find **properties** of the optimal policy function such as monotonicity, continuity, and existence.
- DP can have **computational** advantages. There are methods for numerically approximating policy functions.

Competitive Equilibrium

- We show that the CE allocation coincides with the planner's solution.
- Preferences and technology are the same as before.
- There are two types of agents: households and firms.

Households

- A single representative household owns the capital and rents it to firms at rental rate q.
- It supplies one unit of labor to the firm at wage rate w.
- Preferences are

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

- The budget constraint is:

$$k_{t+1} = (1-\delta)k_t + w_t + q_t k_t - c_t$$

Households: DP Representation

- State variable: k.
- Control: k'.
- Bellman equation:

$$V(k) = \max u([1-\delta]k + w + qk - k') + \beta V(k')$$

- FOC

$$u'(c) = \beta V'(k')$$

- Envelope:

$$V'(k) = u'(c)(1 - \delta + q)$$

- Euler equation:

$$u'(c) = \beta(1 + q' - \delta)u'(c')$$

Household: Solution

A pair of policy functions $c = \phi(k)$ and $k' = h(k)$ and a value function such that:

1. the policy functions solve the "max" part of the Bellman equation, given V;
2. the value function satisfies

$$V(k) = u([1-\delta] + w + qk - h(k)) + \beta \ V(h(k))$$

In terms of sequences: $\{c_t, k_{t+1}\}$ that solve the Euler equation and the budget constraint.

The boundary conditions are k_0 given and the transversality condition (TVC)

$$\lim_{t\to\infty} \beta^t u'(c_t) k_t = 0$$

Firms

- Firms rent capital and labor services from households, taking prices (q, w) as given.
- They maximize current period profits:

$$\max G(K, L) - wL - qK$$

- FOC

$$\begin{aligned} G_K(K, L) &= q \\ G_L(K, L) &= w \end{aligned}$$

Firms

- Assume constant returns to scale. Define

$$g(k^F)L = G(K,L)$$

- FOC's become

$$\begin{aligned} g'(k^F) &= q \\ g(k^F) - g'(k^F)k^F &= w \end{aligned}$$

- A **solution** is a pair (K,L) that satisfies the 2 FOC.

Equilibrium

An equilibrium is a sequence of prices $\{q_t, w_t\}$ and quantities $\{c_t, k_t, L_t\}$ that satisfy:

1. 2 household equations
2. 2 firm FOCs
3. 3 market clearing conditions (one redundant).

Market clearing:

1. Labor: $L_t = 1$.
2. Capital: $k_t = k_t^F$.
3. Goods: $k_{t+1} + c_t = g(k_t) + (1-\delta)k_t$.

Comparison with the Planner's Solution

One way of showing that the Planner's solution coincides with the CE is to appeal to the First and Second Welfare Theorems.

A more direct way is to show that the equations that characterize CE and the planner's solution are the same.

CE
$u'(c) = (1 + q' - \delta)\ \beta\ u'(c')$
$k' + c = g(k) + (1 - \delta)\ k$
$k' = (1 - \delta) k + w + qk - c$
$q = g'(k)$
$w = g(k) - g'(k) k$

Planner
$u'(c) = (g'(k') + 1 - \delta)\ \beta\ u'(c')$
$k' + c = g(k) + (1 - \delta)\ k$

Recursive competitive equilibrium

Recursive CE is an alternative way of representing a CE that is more fully consistent with the DP approach.

- Everything is written as functions of the state variables.
- There are no sequences.

This is useful especially in models with

- heterogeneous agents where the distribution of households is a state variable;
- uncertainty, where we cannot assume that agents take future prices as given.

Recursive competitive equilibrium

- By definition, everything in the economy is a function of the state variables.
- All the agents need to know are the **laws of motion** for the state variables.
  - E.g., to form expectations over future interest rate, use the law of motion for k and the price function $q = f'(k)$.
- Agents' policy functions depends on the laws of motion.
- The laws of motion depend on agents' policy functions.

RCE in the example

- The economy's *state variable* is K.
- Call its law of motion $K' = \varphi(K)$.
  - This is part of the equilibrium.
- We solve the household problem for a saving function $k' = h(k, K)$.
  - It depends on the private state k and the aggregate state K.
- We solve the firm's problem for price functions $q(K), w(K)$.

Household

The household solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$k_{t+1} = (1-\delta)k_t + w(K_t) + q(K_t)k_t - c_t$$

The household's problem has an individual state k and an aggregate state K.

Household

Bellman's equation is

$$\begin{aligned} V(k,K) &= \max u([1-\delta]k + w(K) + q(K)k - k') + \beta V(k',K') \\ K' &= \varphi(K) \end{aligned}$$

Solution: $k' = h(k,K)$.

Firm

Nothing changes in the firm's problem.

Solution:

$$\begin{aligned} q(K) &= g'(K) \\ w(K) &= g(K) - g'(K)K \end{aligned}$$

RCE:

Objects:

- price functions $[q(K), w(K)]$,
- a law of motion for the aggregate state: $K' = \varphi(K)$,
- a policy function $k' = h(k, K)$ and a value function $V(k, K)$.

Equilibrium conditions:

- Given $\varphi(K), q(K), w(K)$: the policy function solves the household's DP.
- The price functions satisfy firm FOCs.
- Markets clear (same as before).
- Household expectations are consistent with household behavior:

$$h(K, K) = \varphi(K)$$

- There are N_j households of type j.
- The representative type j household solves:

$$\begin{aligned} & \max \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \\ s.t.\ k_{t+1} &= R_t k_t + w_t l_t - c_t \end{aligned}$$

Aggregate State

- The aggregate state vector is the distribution of wealth:

$$\kappa = (\kappa_1, ..., \kappa_N) \tag{10}$$

- κ_j is wealth of household j in equilibrium.
- The household knows the law of motion

$$\kappa' = \phi(\kappa) \tag{11}$$

with jth element

$$\kappa'_j = \phi_j(\kappa) \tag{12}$$

Household Dynamic Program

$$
\begin{aligned}
V_j(k_j, \kappa) &= \max u(c_j, l_j) + \beta V_j\left(k'_j, \phi(\kappa)\right) \\
k'_j &= R(\kappa) k_j + w(\kappa) l_j - c_j
\end{aligned}
$$

First-order conditions:

$$
\begin{aligned}
u_c(c_j, l_j) &= \beta V_{j,1}\left(k'_j, \phi(\kappa)\right) && (13) \\
u_l(c_j, l_j) &= \beta V_{j,1}\left(k'_j, \phi(\kappa)\right) \; w(\kappa) && (14)
\end{aligned}
$$

Envelope:

$$
V_{j,1}(k_j, \kappa) = u_c(c_j, l_j) \; R(\kappa) \qquad (15)
$$

Household solution

A solution to the type j household problem consists of

- a value function V_j
- policy functions $k_j' = h_j(k_j, \kappa)$, $l_j = \ell_j(k_j, \kappa)$, and $c_j = g_j(k_j, \kappa)$.

These satisfy:

1. V_j is a fixed point of the Bellman equation, given h, ℓ and g.
2. h, ℓ and g "max" the Bellman equation.

Firm

- This is standard:

$$\max F(K(\kappa), L(\kappa)) - w(\kappa) L(\kappa) - q(\kappa) K(\kappa)$$

- FOC: Factor prices equal marginal products.
- Solution: $K(\kappa)$ and $L(\kappa)$.

Market clearing

- Goods:

$$F(K(\kappa), L(\kappa)) + (1-\delta) K(\kappa) = \sum_j N_j \left[g_j(\kappa_j, \kappa) + h_j(\kappa_j, \kappa) \right] \quad (16)$$

- Labor:

$$L(\kappa) = \sum_j N_j \ell(\kappa_j, \kappa) \quad (17)$$

- Capital:

$$K(\kappa) = \sum_j N_j \kappa_j \quad (18)$$

Recursive CE

- Objects: Functions V_j, h_j, ℓ_j, g_j and $K(\kappa), L(\kappa), w(\kappa), q(\kappa), R(\kappa)$ and ϕ.
- These satisfy:
  1. Household solution (4)
  2. Firm first order conditions (2)
  3. Market clearing (3 - 1 redundant)
  4. Identity: $R(\kappa) = q(\kappa) + 1 - \delta$.
  5. Consistency:
  $$\phi_j(\kappa) = h_j(\kappa_j, \kappa) \ \forall j \tag{19}$$

Notes on RCE

All the objects to be found are functions, not sequences.

This helps when there are shocks:

- We cannot find the sequence of prices without knowing the realizations of the shocks.
- But we can find how prices evolve for each possible sequence of shocks.
- The price functions describe this together with the laws of motion for the states.

Notes on RCE

Functional analysis helps determine the properties of the policy functions and the laws of motion.

- E.g., we strictly concave utility we know that savings are increasing in k, continuous, differentiable, etc.

RCE helps compute equilibria.

- Find the household's optimal choices for every possible set of states.
- Then simulate household histories to find the laws of motion.

Reading

- Acemoglu, "Introduction to Modern Economic Growth," ch. 6. Also ch. 5 for background material we will discuss in detail later on.
- Sargent & Ljunqvist, ch. 3 (Dynamic Programming), ch. 7 (Recursive CE).
- Stokey and Lucas, ch. 1 is a nice introduction.
- Blanchard & Fischer (1989) have a good introduction to the standard growth model.