

1 Land Prices with Capital Accumulation

Consider the following economy with land and capital. There is a single good produced from land L and capital K according to

$$K_{t+1} = AF(K_t, L_t) + (1 - \delta)K_t - c_t \quad (1)$$

where A is an exogenous productivity factor, δ is the depreciation rate of capital, and c is consumption. The production function has constant returns to scale. Production takes place in a representative firm which rents capital and land from households. The rental prices are r_t and q_t . The purchase price of land is p_t . The aggregate endowment of land, L , is fixed.

A single representative household orders consumption sequences according to $\sum_{t=0}^{\infty} \beta^t u(c_t)$. He receives income only from holding capital and land.

- (a) Set up the household's Bellman equation. Define a solution to the household problem.
- (b) Define a competitive equilibrium.
- (c) Determine the effects of the following changes on steady state prices and quantities. A qualitative characterization is sufficient (which variables increase/decrease?): L increases, A increases.

2 Wealth in the utility function

Consider the following modification of the standard growth model where the households derives utility from holding wealth. The representative household solves

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1})$$

subject to the budget constraint

$$k_{t+1} = R_t k_t + w_t - c_t$$

The utility function is strictly concave and increasing in both arguments. The representative firm rents capital and labor from the household at rental prices q_t and w_t to maximize period profits. The production function is given by $F(K_t, L_t)$ and has standard properties including Inada conditions and constant returns to scale.

- (a) State the household problem.
- (b) Derive and explain the conditions that characterize a solution to the household problem (in sequence language).
- (c) Define a competitive equilibrium.
- (d) Derive a single equation that determines the steady state capital stock.
- (e) Is the steady state unique? Explain the intuition why the steady state is or is not unique.

3 Answers

3.1 Answer: Land Prices with Capital Accumulation

(a) Since the household's portfolio composition will be indeterminate, it can be set up with a single asset a with gross return R . This is standard and leads to an Euler equation $u'(c) = \beta R' u'(c')$. The budget constraint is $a' = R a - c$.

An alternative is to set up a problem with two assets. The budget constraint is then $k' + p l' + c = (r + 1 - \delta) k + (p + q) l$. The Bellman equation is

$$V(k, l) = \max u((r + 1 - \delta) k + (p + q) l - k' - p l') + \beta V(k', l') \quad (2)$$

The first-order conditions are

$$\begin{aligned} u'(c) &= \beta V_k(k', l') \\ u'(c) p &= \beta V_l(k', l') \end{aligned} \quad (3)$$

The envelope conditions are

$$\begin{aligned} V_k(k, l) &= u'(c) (r + 1 - \delta) \\ V_l(k, l) &= u'(c) (p + q) \end{aligned}$$

Combining those yields the Euler equation and the arbitrage condition $r' + 1 - \delta = (p' + q')/p$ which says that both assets must yield the same rate of return.

(b) A competitive equilibrium is a set of sequences $(c_t, a_t, K_t, R_t, r_t, q_t, p_t)$ which satisfy

- Household: Euler equation and budget constraint.
- Firms: $r_t = A f'(k_t)$ and $q_t = A [f(k_t) - f'(k_t) k_t]$ where $k_t = K_t/L_t$.
- Identities: $a_t = K_t + p_t L_t$ and $R_t = 1 - \delta + r_t$.
- Goods market clearing (1).
- Asset market clearing: $R_t = (p_t + q_t)/p_{t-1}$.

(c) The steady state is characterized by a recursive system. $R = 1/\beta$. k is determined from $r = R - 1 + \delta = f'(k)$. Then q follows from the firm's first-order condition. Market clearing implies $c = L[A f(k) - \delta k]$. Finally, the price of land equals $p = q/(R - 1)$.

An increase in L has no effect on R, k, q, p . c and K rise in proportion to L . The intuition is that the economy has constant returns to scale. Increasing the fixed factor simply raises all real variables in proportion, but leaves prices unaffected.

An increase in A has no effect on R . Hence, $f'(k)$ must fall and k must rise. This in turn implies a higher q . Therefore p rises. From market clearing, consumption increases.

3.2 Answer: Wealth in the utility function

(a) Bellman equation

$$V(k, z) = \max u(c, z) + \beta V(k', k) + \lambda [Rk + w - c - k']$$

where $z' = k$. Controls are c, k' .

Alternative: Lagrangian:

$$\sum_{t=0}^{\infty} \beta^t u(c_t, k_{t-1}) + \lambda_t [R_t k_t + w_t - c_t - k_{t+1}]$$

(b) First-order conditions from DP:

$$\begin{aligned} u_c(c, z) &= \lambda \\ \beta V_k(k', k) &= \lambda \end{aligned}$$

Envelope conditions:

$$\begin{aligned} V_k(k, z) &= \beta V_z(k', z') + \lambda R \\ V_z(k, z) &= u_z(c, z) \end{aligned}$$

Euler equation:

$$u_c(c, z) = \beta R' u_c(c', z') + \beta^2 u_z(c'', z'')$$

Solution: Sequences (c_t, k_t) that solve the Euler equation, budget constraint, and transversality condition. Intuition: The additional u_z term in the Euler equation reflects the effect of raising k' in the usual perturbation.

Alternative: Lagrangian FOCs:

$$\begin{aligned} c_t &: \beta^t u_c(t) = \lambda_t \\ k_{t+1} &: \beta^{t+2} u_k(t+2) = \lambda_t + \lambda_{t+1} R_{t+1} \end{aligned}$$

(c) CE: Totally standard. $\{c_t, k_{t+1}, w_t, R_t\}$ that satisfy: Household Euler equation and budget constraint. 2 firm first-order conditions. Market clearing: $K_t = k_t, L_t = 1, F(K_t, L_t) + (1 - \delta) K_t = c_t + K_{t+1}$.

(d) The Euler equation implies in steady state:

$$1 = \beta [f'(k) + 1 - \delta] + \beta^2 \frac{u_z(f(k) - \delta k, k)}{u_c(f(k) - \delta k, k)}$$

(e) Steady state is generally not unique. Household may choose low c and high k or vice versa.