

Dynamic Programming

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Econ720

July 26, 2011

Introduction to Dynamic Programming

- Useful theorems to characterize the solution to a DP problem.
- There is no reason to remember these results.
- But you need to know they exist and can be looked up when you need them.

Problem P1: (The sequence problem)

$$V^*(x(0)) = \max_{\{x(t+1)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1))$$

subject to

$$x(t+1) \in G(x(t))$$

$x(0)$ given

$x(t) \in X \subset \mathbb{R}^k$ is the set of allowed states.

The correspondence $G : X \rightrightarrows X$ defines the constraints.

Generic problem

Assumptions that could be relaxed at a cost

- ① Stationarity: U and G do not depend on t .
- ② Utility is additively separable.
 - Time consistency
- ③ The control is $x(t+1)$.
 - There could be additional controls that don't affect $x(t+1)$.
 - They are "max'd out". Ex: 2 consumption goods.

Mapping into the growth model

$$\begin{aligned} & \max_{\{k(t+1), c(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(f(k(t)) - k(t+1)) \\ \text{subject to} & \quad k(t+1) \in [0, f(k(t))] \\ & \quad k(0) \text{ given} \end{aligned}$$

Mapping into the growth model

Define

$$U(k(t), k(t+1)) = u(k(t+1) - f(k(t)))$$

Define

$$G(k(t)) = \{k(t+1) : k(t+1) \in [0, f(k(t))]\}$$

Problem P2:

$$V(x) = \max_{y \in G(x)} U(x, y) + \beta V(y), \forall x \in X$$

This is a Bellman equation.

The question: When is solving P1 equivalent to solving P2?

A solution is a policy function $\pi : X \rightarrow X$ and a value function $V(x)$ such that

- 1 $V(x) = U(x, \pi(x)) + \beta V(\pi(x)), \forall x \in X$
- 2 When $y = \pi(x)$, now and forever, the max value is attained.

Dynamic Programming Theorems

- The payoff of DP: it is easier to prove that solutions exist, are unique, monotone, etc.
- We state some assumptions and theorems using them.

Assumption 1

- Define the set of feasible paths starting at $x(0)$ by $\Phi(x(0))$.
- $G(x)$ is nonempty for all $x \in X$.
 - needed to prevent a currently good looking path from running into "dead ends"
- $\lim_{n \rightarrow \infty} \sum_{t=0}^n \beta^t U(x(t), x(t+1))$ exists and is finite, for all $x(0) \in X$ and feasible paths $x \in \Phi(x(0))$.
 - cannot have unbounded utility

Assumption 2

- The set X in which x lives is compact.
- G is compact valued and continuous.
- U is continuous.

Notes:

- Compactness avoids existence issues: without it, there could always be a slightly better x
- Compact X creates trouble with endogenous growth, but can be relaxed.

Assumption 3

- U is strictly concave.
- G is convex (for all x , $G(x)$ is a convex set).

Typical assumptions to ensure that first order conditions are sufficient.

Assumption 4

- $U(x, y)$ is strictly increasing in x .
 - more capital is better
- G is monotone in the sense that $x \leq x'$ implies $G(x) \subset G(x')$.

This is needed for **monotonicity** of policy function.

Assumption 5

- U is continuously differentiable on the interior of its domain.

So we can work with first-order conditions.

Theorem 1: Equivalence of values

- Assume A1 and A2.
- Then for any x , $V^*(x) = V(x)$.
 - The value that comes out of solving the sequence problem also solves P2.
 - Solving P2 means: Stick V^* into P2 and max. Then V^* pops out on the LHS.
- And any $V(x)$ that solves P2 and satisfies $\lim_{t \rightarrow \infty} \beta^t V(x(t)) = 0$ for all feasible x satisfies $V(x) = V^*(x)$.

Theorem 1: Equivalence of values

In words:

- For any initial x , P1 and P2 yield the same values.
- This says nothing about the policies.

Theorem 2: Principle of Optimality

- Assume A1.
- In P1, for any **optimal** plan \mathbf{x}^* [that attains $V^*(\mathbf{x}(t))$ in P1] starting at $\mathbf{x}(0)$ the Bellman equation holds:

$$V^*(\mathbf{x}^*(t)) = U(\mathbf{x}^*(t), \mathbf{x}^*(t+1)) + \beta V^*(\mathbf{x}^*(t+1)) \quad (1)$$

- Any feasible plan \mathbf{x}^* starting at $\mathbf{x}(0)$ that satisfies (1) attains the max value in P1.

Theorem 2: Principle of Optimality

In words:

- Solve the sequence problem to get V^* and x^* . Both satisfy the Bellman equation (without the max part).
- Part 2 gives us the max part: If (1) holds for x^* , then x^* solves the max part.
- If we solve the sequence problem, we solve the recursive one.

Theorem 2: Principle of Optimality

- Part 2 says: we can go the other way.
- Solve the Bellman equation to get $V(x)$ and optimal sequences x^* .
- They satisfy the Bellman equation
- By theorem 1, they also satisfy the Bellman equation with value $V^*(x)$.
- Part 2 says: x^* then also solves the sequence problem.

Theorem 2: Principle of Optimality

- To sum up: If A1 and A2 hold, then solving the sequence problem and solving the recursive problem yield the same values and policies.

Theorem 3: Uniqueness of V

- Assumptions: A1 and A2.
- Then there exists a unique, **continuous**, **bounded** value function that solves P1 or P2 (they are the same).
- An optimal plan x^* exists. But it may not be unique.

Theorem 4: Concavity of V

- Assumptions: A1-A3.
- Then the value function is strictly concave.

Recall: A3 says that U is strictly concave and $G(x)$ is convex.

Corollary 1

- Assumptions A1-A3.
- Then there exists a unique optimal plan \mathbf{x}^* for all $\mathbf{x}(0)$.
- It can be written as $\mathbf{x}^*(t+1) = \boldsymbol{\pi}(\mathbf{x}^*(t))$.
- $\boldsymbol{\pi}$ is continuous.

Reason: The Bellman equation is a concave optimization problem with convex choice set.

Theorem 5: Monotonicity of V

- Assumptions: A1, A2, A4.
- Recall A4: U and G are monotone.
- V is strictly increasing in all arguments (states).

Theorem 6: Differentiability of V

- Assumptions A1, A2, A3, A5.
- A5: U is differentiable.
- Then $V(x)$ is continuously differentiable at all interior points x' with $\pi(x') \in \text{Int}G(x')$.
- The derivative is given by:

$$DV(x') = D_x U(x', \pi(x')) \quad (2)$$

This is an envelope condition: we can ignore the response of π when x' changes.

Contraction mapping theorem

- How could one show that V is increasing? Or concave? Etc.
- Thinking of the Bellman equation as a functional equation helps...
- Think of the Bellman equation as mapping V on the RHS into \hat{V} on the LHS:

$$\hat{V}(x) = \max_{y \in G(x)} U(x, y) + \beta V(y) \quad (3)$$

- The RHS is a function of V .
- The Bellman equation maps the space of functions V lives in into itself.

$$\hat{V} = T(V) \quad (4)$$

- The solution is the function V that is a fixed point of T :

$$V = T(V) \quad (5)$$

- If $T : X \rightarrow X$, we write:
 - 1 Tx instead of the usual $T(x)$
 - 2 $T(\hat{X})$ as the image of the set $\hat{X} \subset X$.

Contraction mapping theorem

- The Bellman equation is $\hat{V} = TV$.
- Suppose we could show:
 - 1 If V is increasing, then \hat{V} is increasing.
 - 2 There is a fixed point in the set of increasing functions.
 - 3 The fixed point is unique.
- Then we would have shown that the solution V is increasing.
- The contraction mapping theorem allows us to make arguments like this.

Contraction mapping theorem

Definition

Let (S, d) be a metric space and $T : S \rightarrow S$. T is a contraction mapping with modulus β , if for some $\beta \in (0, 1)$,

$$d(Tz_1, Tz_2) \leq \beta d(z_1, z_2), \quad \forall z_1, z_2 \in S \quad (6)$$

A contraction pulls points closer together.

Contraction mapping theorem

Theorem 7: Let (S, d) be a complete metric space and let T be a contraction mapping. Then T has a unique fixed point in S .

Contraction mapping theorem

A helpful result for showing properties of V :

*Theorem 8: Let (S, d) be a complete metric space and let $T : S \rightarrow S$ be a contraction mapping with fixed point $T\hat{z} = \hat{z}$.
If S' is a closed subset of S and $T(S') \subset S'$, then $\hat{z} \in S'$.
If $T(S') \subset S'' \subset S'$, then $\hat{z} \in S''$.*

point: When looking for the fixed point, one can restrict the search to sub-spaces with nice properties.

Example: V may be defined on a broad set of functions. But if one can show that T maps strictly increasing function into themselves, then the fixed point must be strictly increasing.

Blackwell's Sufficient Conditions

This is helpful for showing that a Bellman operator is a contraction:

Theorem 9: Let $X \subseteq \mathbb{R}^K$, and $\mathbf{B}(X)$ be the space of bounded functions $f : X \rightarrow \mathbb{R}$. Suppose that $T : \mathbf{B}(X) \rightarrow \mathbf{B}(X)$ satisfies:

(1) monotonicity: $f(x) \leq g(x)$ for all $x \in X$ implies $Tf(x) \leq Tg(x)$ for all $x \in X$.

(2) discounting: there exists $\beta \in (0, 1)$ such that

$$T[f(x) + c] \leq Tf(x) + \beta c \text{ for all } f \in \mathbf{B}(X) \text{ and } c \geq 0.$$

Then T is a contraction with modulus β .

need examples +++

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Summary: Contraction mapping theorem

Suppose you want to show that the value function is increasing.

- 1 Show that the Bellman equation is a contraction mapping - using Blackwell.
- 2 Show that it maps increasing functions into increasing functions.

Done.

First order conditions

Consider again Problem P2:

$$V(x) = \max_{y \in G(x)} U(x, y) + \beta V(y), \quad \forall x \in X$$

If we make assumptions that ensure:

- V is differentiable and concave.
- U is concave.
- G is convex. [A1-A5 ensure all that.]

Then the RHS is just a standard concave optimization problem.

We can take the usual FOCs to characterize the solution.

First order conditions

- For y :

$$D_y U(x, \pi(x)) + \beta DV(\pi(x)) = 0 \quad (7)$$

- To find $DV(x)$ differentiate the Bellman equation:

$$DV(x) = D_x U(x, \pi(x)) + D_y U(x, \pi(x)) D\pi(x) + \beta DV(\pi(x)) D\pi(x) = 0 \quad (8)$$

- Apply the FOC to find the Envelope condition:

$$DV(x) = D_x U(x, \pi(x)) \quad (9)$$

$$DV(\pi(x)) = D_x U(\pi(x), \pi(\pi(x))) \quad (10)$$

- Sub back into the FOC:

$$D_y U(x, \pi(x)) + \beta D_x U(\pi(x), \pi(\pi(x))) = 0 \quad (11)$$

First order conditions

- In the usual prime notation:

$$D_2 U(x, x') + \beta D_1 U(x', x'') = 0 \quad (12)$$

- Think about a feasible perturbation:
 - 1 Raise x' a little and gain $D_2 U(x, x')$ today.
 - 2 Tomorrow lose the marginal value of the state x' : $D_1(x', x'')$.
- Why isn't there a term as in the growth model's resource constraint:
 $f'(k) + 1 - \delta$?
 - By writing $U(x, x')$, the resource constraint is built into U .
 - In the growth model: $U(k, k') = u(f(k) + (1 - \delta)k - k')$.
 - $D_1 U = u'(c)[f'(k) + 1 - \delta]$.

- Even though the programming problem is concave, the first-order condition is not sufficient!
- A mechanical reason: it is a first-order difference equation - it has infinitely many solutions.
- A boundary condition is needed.

Theorem 10: Let $X \subset \mathbb{R}^K$ and assume A1-A5. Then a sequence $\{x(t+1)\}$ with $x(t+1) \in \text{Int}G(x(t))$ is optimal in P1, if it satisfies the Euler equation and the transversality condition

$$\lim_{t \rightarrow \infty} \beta^t D_x U(x(t), x(t+1)) x(t) = 0 \quad (13)$$

Example: The growth model

$$\max \sum_{t=0}^{\infty} \beta^t \ln(c(t))$$

subject to

$$0 \leq k(t+1) \leq k(t)^\alpha - c(t)$$

$$k(0) = k_0$$

Example: The growth model

- Step 1: Show that A1 to A5 hold.
- Define $U(k, k') = \ln(k^\alpha - k')$.
- A1 is obvious: $G(x)$ is non-empty. The sum of discounted utilities is bounded for all feasible paths.
- A2:
 - X is compact - no, but we can restrict k to a compact set w.l.o.g.
 - G is compact valued and continuous: check
 - U is continuous: check
- A3: U is strictly concave. $G(x)$ is convex: check.
- A4: U is strictly increasing in x . G is monotone: check.
- A5: U is continuously differentiable: check

Example: The growth model

- Step 2: Theorems 1-6 and 10 apply.
- We can characterize the solution by first-order conditions and TVC.
- FOC:

$$\frac{1}{k^\alpha - \pi(k)} = \beta V'(\pi(k)) \quad (14)$$

- Envelope:

$$V'(k) = \frac{\alpha k^{\alpha-1}}{k^\alpha - \pi(k)} \quad (15)$$

- Combine:

$$\frac{1}{k^\alpha - \pi(k)} = \beta \frac{\alpha \pi(k)^{\alpha-1}}{\pi(k)^\alpha - \pi(\pi(k))} \quad (16)$$

- Or:

$$u'(c) = \beta f'(k') u'(c') \quad (17)$$

Example: The growth model

Other things we know:

- 1 V is continuously differentiable, bounded, unique, strictly concave.
- 2 $V'(k) > 0$.
- 3 The optimal policy function $c = \phi(k)$ is unique, continuous.

- Acemoglu, *Introduction to Modern Economic Growth*, ch. 6
- Stokey, Lucas, with Prescott, *Recursive Methods*. A book length treatment. The standard reference.