

# Dynamic Programming

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# Introduction to Dynamic Programming

- Useful theorems to characterize the solution to a DP problem.
- There is no reason to remember these results.
- But you need to know they exist and can be looked up when you need them.

Problem P1: (The sequence problem)

$$\begin{aligned} V^*(x(0)) &= \max_{\{x(t+1)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t U(x(t), x(t+1)) \\ &\text{subject to} \\ x(t+1) &\in G(x(t)) \\ x(0) &\text{ given} \end{aligned}$$

$x(t) \in X \subset \mathbb{R}^k$  is the set of allowed states.

The correspondence  $G : X \rightrightarrows X$  defines the constraints.

# Generic problems

Assumptions that could be relaxed at a cost

- 1 Stationarity:  $U$  and  $G$  do not depend on  $t$ .
- 2 Utility is additively separable.
  - Time consistency

# Mapping into the growth model

$$\max_{\{k(t+1), c(t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c(t))$$

*subject to*

$$k(t+1) = f(k(t)) - c(t) \geq 0$$

$k(0)$  given

# Mapping into the growth model

Define

$$U(k(t), k(t+1)) = u(k(t+1) - f(k(t)))$$

Define

$$G(k(t)) = \{k(t+1) : k(t+1) \in [0, f(k(t))]\}$$

Problem P2:

$$V(x) = \max_{y \in G(x)} U(x, y) + \beta V(y), \quad \forall x \in X$$

This is a Bellman equation.

The question: When is solving P1 equivalent to solving P2?

A solution is a policy function  $\pi : X \rightarrow X$  and a value function  $V(x)$  such that

$$V(x) = U(x, \pi(x)) + \beta V(\pi(x)), \forall x \in X$$

When  $y = \pi(x)$ , now and forever, the max value is attained.

# Dynamic Programming Theorems

- The payoff of DP: it is easier to prove that solutions exist, are unique, monotone, etc.
- We state some assumptions and theorems using them.

# Assumption 1

- Define the set of feasible path starting at  $x(0)$  by  $\Phi(x(0))$ .
- $G(x)$  is nonempty for all  $x \in X$ .
  - needed to prevent a currently good looking path from running into "dead ends"
- $\lim_{n \rightarrow \infty} \sum_{t=0}^n \beta^t U(x(t), x(t+1))$  exists and is finite, for all  $x(0) \in X$  and feasible paths  $\mathbf{x} \in \Phi(x(0))$ .
  - cannot have unbounded utility

# Assumption 2

- The set  $X$  in which  $x$  lives is compact.
- $G$  is compact valued and continuous.
- $U$  is continuous.

## Notes:

- Compactness avoids existence issues: without it, there could always be a slightly better  $x$
- Compact  $X$  creates trouble with endogenous growth, but can be relaxed.

# Assumption 3

- $U$  is strictly concave.
- $G$  is convex (for all  $x$ ,  $G(x)$  is a convex set).

Typical assumptions to ensure that first order conditions are sufficient.

# Assumption 4

- $U(x, y)$  is strictly increasing in  $x$ .
  - more capital is better
- $G$  is monotone in the sense that  $x \leq x'$  implies  $G(x) \subset G(x')$ .

This is needed for **monotonicity** of policy function.

# Assumption 5

- $U$  is continuously differentiable on the interior of its domain.

So we can work with first-order conditions.

# Theorem 1: Equivalence of values

- Assume A1 and A2.
- Then for any  $x$ ,  $V^*(x) = V(x)$ .
  - The value that comes out of solving the sequence problem also solves P2.
  - Solving P2 means: Stick  $V^*$  into P2 and max. Then  $V^*$  pops out on the LHS.
- And any  $V(x)$  that solves P2 and satisfies  $\lim_{t \rightarrow \infty} \beta^t V(x(t)) = 0$  for all feasible  $x$  satisfies  $V(x) = V^*(x)$ .

# Theorem 1: Equivalence of values

In words:

- For any initial  $x$ , P1 and P2 yield the same values.
- This says nothing about the policies.

## Theorem 2: Principle of Optimality

- Assume A1.
- In P1, for any **optimal** plan  $\mathbf{x}^*$  [that attains  $V^*(x(t))$  in P1] starting at  $x(0)$  the Bellman equation holds:

$$V^*(x^*(t)) = U(x^*(t), x^*(t+1)) + \beta V^*(x^*(t+1)) \quad (1)$$

- Any feasible plan  $\mathbf{x}^*$  starting at  $x(0)$  that satisfies (1) attains the max value in P1.

## Theorem 2: Principle of Optimality

In words:

- Solve the sequence problem to get  $V^*$  and  $\mathbf{x}^*$ . Both satisfy the Bellman equation (without the max part).
- Part 2 gives us the max part: If (1) holds for  $\mathbf{x}^*$ , then  $\mathbf{x}^*$  solves the max part.
- If we solve the sequence problem, we solve the recursive one.

## Theorem 2: Principle of Optimality

- Part 2 says: we can go the other way.
- Solve the Bellman equation to get  $V(x)$  and optimal sequences  $\mathbf{x}^*$ .
- They satisfy the Bellman equation
- By theorem 1, they also satisfy the Bellman equation with value  $V^*(x)$ .
- Part 2 says:  $\mathbf{x}^*$  then also solves the sequence problem.

## Theorem 2: Principle of Optimality

- To sum up: If A1 and A2 hold, then solving the sequence problem and solving the recursive problem yield the same values and policies.

## Theorem 3: Uniqueness of $V$

- Assumptions: A1 and A2.
- Then there exists a unique, **continuous**, **bounded** value function that solves P1 or P2 (they are the same).
- An optimal plan  $x^*$  exists. But it may not be unique.

## Theorem 4: Concavity of $V$

- Assumptions: A1-A3.
- Then the value function is strictly concave.

Recall: A3 says that  $U$  is strictly concave and  $G(x)$  is convex.

# Corollary 1

- Assumptions A1-A3.
- Then there exists a unique optimal plan  $\mathbf{x}^*$  for all  $x(0)$ .
- It can be written as  $x^*(t+1) = \pi(x^*(t))$ .
- $\pi$  is continuous.

Reason: The Bellman equation is a concave optimization problem with convex choice set.

# Theorem 5: Monotonicity of $V$

- Assumptions: A1, A2, A4.
- Recall A4:  $U$  and  $G$  are monotone.
- $V$  is strictly increasing in all arguments (states).

## Theorem 6: Differentiability of $V$

- Assumptions A1, A2, A3, A5.
- A5:  $U$  is differentiable.
- Then  $V(x)$  is continuously differentiable at all interior points  $x'$  with  $\pi(x') \in \text{Int}G(x')$ .
- The derivative is given by:

$$DV(x') = D_x U(x', \pi(x')) \quad (2)$$

This is an envelope condition: we can ignore the response of  $\pi$  when  $x'$  changes.

# Contraction mapping theorem

- How could one show that  $V$  is increasing? Or concave? Etc.
- Thinking of the Bellman equation as a functional equation helps...
- Think of the Bellman equation as mapping  $V$  on the RHS into  $\hat{V}$  on the LHS:

$$\hat{V}(x) = \max_{y \in G(x)} U(x, y) + \beta V(y) \quad (3)$$

- The RHS is a function of  $V$ .
- The Bellman equation maps the space of functions  $V$  lives in into itself.

$$\hat{V} = T(V) \quad (4)$$

- The solution is the function  $V$  that is a fixed point of  $T$ :

$$V = T(V) \quad (5)$$

- If  $T : X \rightarrow X$ , we write:
  - 1  $Tx$  instead of the usual  $T(x)$
  - 2  $T(\hat{X})$  as the image of the set  $\hat{X} \subset X$ .

# Contraction mapping theorem

- The Bellman equation is  $\hat{V} = TV$ .
- Suppose we could show:
  - 1 If  $V$  is increasing, then  $\hat{V}$  is increasing.
  - 2 There is a fixed point in the set of increasing functions.
  - 3 The fixed point is unique.
- Then we would have shown that the solution  $V$  is increasing.
- The contraction mapping theorem allows us to make arguments like this.

# Contraction mapping theorem

## Definition

Let  $(S, d)$  be a metric space and  $T : S \rightarrow S$ .  $T$  is a contraction mapping with modulus  $\beta$ , if for some  $\beta \in (0, 1)$ ,

$$d(Tz_1, Tz_2) \leq \beta d(z_1, z_2), \quad \forall z_1, z_2 \in S \quad (6)$$

• A contraction pulls points closer together.

# Contraction mapping theorem

*Theorem 7: Let  $(S, d)$  be a complete metric space and let  $T$  be a contraction mapping. Then  $T$  has a unique fixed point in  $S$ .*

# Contraction mapping theorem

A helpful result for showing properties of  $V$  :

*Theorem 8: Let  $(S, d)$  be a complete metric space and let  $T : S \rightarrow S$  be a contraction mapping with fixed point  $T\hat{z} = \hat{z}$ . If  $S'$  is a closed subset of  $S$  and  $T(S') \subset S'$ , then  $\hat{z} \in S'$ . If  $T(S') \subset S'' \subset S'$ , then  $\hat{z} \in S''$ .*

The point: When looking for the fixed point, one can restrict the search to sub-spaces with nice properties.

Example:  $V$  may be defined on a broad set of functions. But if one can show that  $T$  maps strictly increasing function into themselves, then the fixed point must be strictly increasing.

This is helpful for showing that a Bellman operator is a contraction:

*Theorem 9: Let  $X \subseteq \mathbb{R}^K$ , and  $\mathbf{B}(X)$  be the space of bounded functions  $f : X \rightarrow \mathbb{R}$ . Suppose that  $T : \mathbf{B}(X) \rightarrow \mathbf{B}(X)$  satisfies:*

*(1) monotonicity:  $f(x) \leq g(x)$  for all  $x \in X$  implies  $Tf(x) \leq Tg(x)$  for all  $x \in X$ .*

*(2) discounting: there exists  $\beta \in (0, 1)$  such that*

$$T[f(x) + c] \leq Tf(x) + \beta c \text{ for all } f \in \mathbf{B}(X) \text{ and } c \geq 0.$$

*Then  $T$  is a contraction with modulus  $\beta$ .*

this need examples +++

# Summary: Contraction mapping theorem

Suppose you want to show that the value function is increasing.

- 1 Show that the Bellman equation is a contraction mapping - using Blackwell.
- 2 Show that it maps increasing functions into increasing functions.

Done.

# First order conditions

Consider again Problem P2:

$$V(x) = \max_{y \in G(x)} U(x, y) + \beta V(y), \quad \forall x \in X$$

If we make assumptions that ensure:

- $V$  is differentiable and concave.
- $U$  is concave.
- $G$  is convex. [A1-A5 ensure all that.]

Then the RHS is just a standard concave optimization problem.  
We can take the usual FOCs to characterize the solution.

# First order conditions

- For  $y$ :

$$D_y U(x, \pi(x)) + \beta DV(\pi(x)) = 0 \quad (7)$$

- To find  $DV(x)$  differentiate the Bellman equation:

$$DV(x) = D_x U(x, \pi(x)) + D_y U(x, \pi(x)) D\pi(x) + \beta DV(\pi(x)) D\pi(x) \quad (8)$$

- Apply the FOC to find the Envelope condition:

$$DV(x) = D_x U(x, \pi(x)) \quad (9)$$

$$DV(\pi(x)) = D_x U(\pi(x), \pi(\pi(x))) \quad (10)$$

- Sub back into the FOC:

$$D_y U(x, \pi(x)) + \beta D_x U(\pi(x), \pi(\pi(x))) = 0 \quad (11)$$

- In the usual prime notation:

$$D_2U(x, x') + \beta D_1U(x', x'') = 0 \quad (12)$$

- Think about a feasible perturbation:

- 1 Raise  $x'$  a little and gain  $D_2U(x, x')$  today.
- 2 Tomorrow lose the marginal value of the state  $x'$ :  $D_1(x', x'')$ .

- Why isn't there a term as in the growth model's resource constraint:

$$f'(k) + 1 - \delta?$$

- By writing  $U(x, x')$ , the resource constraint is built into  $U$ .
- In the growth model:  $U(k, k') = u(f(k) + (1 - \delta)k - k')$ .
- $D_1U = u'(c) [f'(k) + 1 - \delta]$ .

- Even though the programming problem is concave, the first-order condition is not sufficient!
- A mechanical reason: it is a first-order difference equation - it has infinitely many solutions.
- A boundary condition is needed.

*Theorem 10: Let  $X \subset \mathbb{R}^K$  and assume A1-A5. Then a sequence  $\{x(t+1)\}$  with  $x(t+1) \in \text{Int}G(x(t))$  is optimal in P1, if it satisfies the Euler equation and the transversality condition*

$$\lim_{t \rightarrow \infty} \beta^t D_x U(x(t), x(t+1)) x(t) = 0 \quad (13)$$

## Example: The growth model

$$\max \sum_{t=0}^{\infty} \beta^t \ln(c(t))$$

*subject to*

$$0 \leq k(t+1) \leq k(t)^\alpha - c(t)$$

$$k(0) = k_0$$

## Example: The growth model

- Step 1: Show that A1 to A5 hold.
- Define  $U(k, k') = \ln(k^\alpha - k')$ .
- A1 is obvious:  $G(x)$  is non-empty. The sum of discounted utilities is bounded for all feasible paths.
- A2:
  - $X$  is compact - no, but we can restrict  $k$  to a compact set w.l.o.g.
  - $G$  is compact valued and continuous: check
  - $U$  is continuous: check
- A3:  $U$  is strictly concave.  $G(x)$  is convex: check.
- A4:  $U$  is strictly increasing in  $x$ .  $G$  is monotone: check.
- A5:  $U$  is continuously differentiable: check

## Example: The growth model

- Step 2: Theorems 1-6 and 10 apply.
- We can characterize the solution by first-order conditions and TVC.
- FOC:

$$\frac{1}{k^\alpha - \pi(k)} = \beta V'(\pi(k)) \quad (14)$$

- Envelope:

$$V'(k) = \frac{\alpha k^{\alpha-1}}{k^\alpha - \pi(k)} \quad (15)$$

- Combine:

$$\frac{1}{k^\alpha - \pi(k)} = \beta \frac{\alpha \pi(k)^{\alpha-1}}{\pi(k)^\alpha - \pi(\pi(k))} \quad (16)$$

- Or:

$$u'(c) = \beta f'(k') u'(c') \quad (17)$$

# Example: The growth model

Other things we know:

- 1  $V$  is continuously differentiable, bounded, unique, strictly concave.
- 2  $V'(k) > 0$ .
- 3 The optimal policy function  $c = \phi(k)$  is unique, continuous.

- Acemoglu, *Introduction to Modern Economic Growth*, ch. 6
- Stokey, Lucas, with Prescott, *Recursive Methods*. A book length treatment. The standard reference.