

# A Model of Growth and Innovation

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- We study a GE model of growth driven by innovation.
- Innovation takes the form of inventing new goods.
- Alternative: Quality ladders.

# A Model of Product Innovation

- Demographics: A representative household.
- Preferences:

$$\int_0^{\infty} e^{-\rho t} \frac{C_t^{1-\theta} - 1}{1-\theta} dt \quad (1)$$

- Production of final goods from intermediates and labor:

$$Y_t = (1 - \beta)^{-1} \left[ \int_0^{N_t} x(v, t)^{1-\beta} dv \right] L^\beta \quad (2)$$

- This is of the Dixit-Stiglitz form: write  $[\int x^{1-\beta} dv]^{\frac{1-\beta}{1-\beta}}$  to see that this is a CES aggregator of  $x$ .
- Production of intermediates from output: each unit of  $x$  requires  $\psi$  units of  $Y$ .

- Resource constraint:

$$C_t + X_t + Z_t = Y_t$$

- $Z$ : R&D investment.
- $X$ : Inputs into the production of  $x$ .

$$\dot{N} = \eta Z_t \quad (3)$$

Think of this as the aggregate (deterministic) outcome of the (stochastic) innovation efforts of many firms.

# Market arrangements

- Final goods and labor markets are competitive.
- The household owns the innovating firms.
- Innovators receive perpetual patents (monopolies) for their varieties.
- Free entry into innovation (ensures zero profits).

$$Y_t = (1 - \beta)^{-1} \left[ \int_0^{N_t} x(v, t)^{1-\beta} dv \right] L^\beta \quad (4)$$

- Maximize period profits by choosing  $L$  and  $x(v, t)$ .
- Normalize the price  $Y$  to 1.
- Profits

$$Y_t - w_t L_t - \int_0^{N_t} p^x(v, t) x(v, t) dv \quad (5)$$

- Demand function (cf. the Dixit Stiglitz discussion):

$$x(v, t) = L p^x(v, t)^{-1/\beta} \quad (6)$$

**Labor demand** is standard:

$$w_t = \beta Y_t / L_t \quad (7)$$

- **Solution** to the firm's problem:  $L_t, x(v, t)$  that satisfy the "2" first-order conditions.

# Intermediate input producers

- Problem after inventing a variety.
- $x$  is produced at constant marginal cost  $\psi$ .
- Maximize present value of profits

$$V(v, t) = \int_t^{\infty} e^{-rs} \pi(v, s) ds \quad (8)$$

- Instantaneous profits are

$$\pi(v, t) = (p^x(v, t) - \psi) x(v, t) \quad (9)$$

- With constant demand elasticity:

$$p^x = \psi / (1 - \beta) \quad (10)$$

- Profits are

$$\pi(v, t) = \psi \frac{\beta}{1 - \beta} x(v, t) \quad (11)$$

- Solution: A constant  $p^x$ .

- Objects:  $C_t, X_t, Z_t, x(v, t), V(v, t), N_t$  and prices  $p^x(v, t), r(t), w(t)$ .
- Conditions:
  - "Everybody maximizes."
  - Markets clear.
  - Innovation effort satisfies a free entry condition:
- **Symmetric equilibrium:**
  - $x, V$  and  $p^x$  do not depend on  $v$ .

# Equilibrium: Characterization

- Normalize  $\psi = 1 - \beta$  so that  $p^x = 1$ .
- Demand for intermediates:  $x(v, t) = L$ .
- Profits:  $\pi = \beta L$ .
- Production function for final goods:

$$Y = \frac{N_t L}{1 - \beta} \quad (12)$$

# Equilibrium: Characterization

- Labor demand:

$$w_t = \beta \frac{Y_t}{L_t} = \frac{\beta}{1 - \beta} N_t \quad (13)$$

- Free entry:

- Spend 1 to obtain  $\eta$  new patents, each valued at  $V(v, t)$

$$\eta V(v, t) = 1 \quad (14)$$

- Total expenditure on intermediates:

$$X_t = N_t p^x x_t = N_t L \quad (15)$$

(the book has an additional  $1 - \beta$  term).

- Household: standard Euler equation

$$\dot{C}/C = \frac{r - \rho}{\theta} \quad (16)$$

- Value of the firm with profits  $\beta L$ :

$$V = \beta L / r \quad (17)$$

- Free entry:

$$\eta \beta L / r = 1 \quad (18)$$

- This is the closed form solution for  $r$ .
- Balanced growth rate then follows from the Euler equation.

$$g = \frac{\eta\beta L - \rho}{\theta}$$

- Larger economies ( $L$ ) grow faster.
- Mechanical reason:
  - Innovation technology is linear in goods.
  - Larger economy  $\rightarrow$  higher  $Y \rightarrow$  higher  $Z \rightarrow$  faster growth.

# No Transition Dynamics

- Period profits  $\pi$  are constant at  $\beta L$ .
- At any moment we need  $\eta V = 1$ .
- Constant  $V$  is only possible with constant  $r$ .
- Intuition: There is a reduced form AK structure.

$$Y_t = \frac{L}{1 - \beta} N_t$$
$$\dot{N}_t = \eta s_z Y_t$$

# Pareto Efficient Allocation

Two distortions prevent efficiency of equilibrium:

- 1 Monopoly pricing.
- 2 Inefficient innovation due to aggregate demand externality.

# Planner's Problem

Solve in two stages:

- 1 Given  $N$ , find optimal static allocation  $x(v, t)$ .
- 2 Given the reduced form production function from #1, find optimal  $Z$ .

# Static Allocation

Given  $N$ , choose  $x(v, t)$  to maximize  $Y - X$ :

$$\max (1 - \beta)^{-1} L^\beta \int_0^{N_t} x(v, t)^{1-\beta} dv - \int_0^{N_t} \psi x(v, t) dv \quad (19)$$

First-order condition (with  $\psi = 1 - \beta$ ):

$$x(v, t) = x = (1 - \beta)^{-1/\beta} L \quad (20)$$

Reduced form production function:

$$Y_t = (1 - \beta)^{-1/\beta} L N_t \quad (21)$$

Net output

$$\begin{aligned} Y - X &= (1 - \beta)^{-1/\beta} LN - (1 - \beta) (1 - \beta)^{-1/\beta} LN \\ &= (1 - \beta)^{-1/\beta} \beta L N \end{aligned} \quad (22)$$

$$\max \int_0^{\infty} e^{-\rho t} \frac{C_t^{1-\theta} - 1}{1-\theta} dt$$

subject to

$$\dot{N} = \eta Z$$

$$Y = (1-\beta)^{-1/\beta} \beta L N = C + Z$$

Or

$$\dot{N} = A N - \eta C \tag{23}$$

$$A = \eta (1-\beta)^{-1/\beta} \beta L \tag{24}$$

$$H = \frac{C^{1-\theta} - 1}{1-\theta} + \mu [AN - \eta C] \quad (25)$$

FOC

$$\partial H / \partial C = C^{-\theta} - \mu \eta = 0 \quad (26)$$

$$\partial H / \partial N = \rho \mu - \dot{\mu} = \mu A \quad (27)$$

The same as in an AK model with

$$A = \eta (1 - \beta)^{-1/\beta} \beta L \quad (28)$$

we have

$$\dot{C}/C = \frac{A - \rho}{\theta} \quad (29)$$

# Comparison with CE

- CE interest rate:  $\eta\beta L$ .
- Planner's "interest rate:"  $(1 - \beta)^{-1/\beta} \eta\beta L$ .
- The planner chooses faster growth.
- Intuition:
  - CE under-utilizes the fruits of innovation:  $x$  is too low.
  - This reduces the value of innovation.

- One might be tempted to reduce monopoly power.
- A policy that encourages competition (e.g. less patent protection, forcing lower  $p^x$ ) reduces the static price distortion.
- But it also reduces growth: innovation is less valuable.
- Similar result for shorter patents.
- Policy trades off static efficiency and incentives for innovation.

# Knowledge Spillovers

# Ideas Produced From Labor

- The previous model had endogenous growth because ideas were produced from a **reproducible factor**: goods.
- If ideas are produced from (non-reproducible) labor: there is no sustained growth.
- As we showed before: sustained growth requires constant returns to reproducible factors.
  - More precisely: returns to reproducible factors must be **asymptotically bounded from below**.

## Example

Assume  $\dot{N}_t = \eta Z_t^\alpha L_{Rt}^{1-\alpha}$ . Show that the balanced growth rate is 0 unless  $\alpha = 1$ .

# Knowledge Spillovers

- If any non-reproducible factors are used in innovation, knowledge spillovers are required to sustain growth.
- Knowledge spillovers means:  $N$  appears in the innovation production function.
- A **knife-edge** parameter assumption is needed for endogenous growth.
- This is always true because we need **constant returns to reproducible factors**.

# Knowledge spillover model

- Production of ideas:

$$\dot{N}_t = \eta N_t L_{Rt} \quad (30)$$

- We show later: linearity in  $N$  is required for endogenous growth.
- Resource constraint:

$$L = L_{Rt} + L_{Et} \quad (31)$$

- Production of final goods (same as before):

$$Y_t = \frac{N_t L_{Et}}{1 - \beta} \quad (32)$$

- Wage rate (unchanged):

$$w_t = \frac{\beta}{1 - \beta} N_t \quad (33)$$

- Profits earned by monopolists:

$$\pi_t = \beta L_{Et} \quad (34)$$

- Value of the firm with profits  $\beta L_{Et}$  in steady state:

$$V = \beta L_E / r \quad (35)$$

- Free entry:

$$\eta N_t V_t = w_t \quad (36)$$

- the same as before but R&D productivity is now  $\eta N$  instead of  $\eta$ .

# Characterizing balanced growth

- Sub wage rate into free entry:

$$\eta N_t \frac{\beta L_E}{r} = \frac{\beta}{1 - \beta} N_t \quad (37)$$

$\implies$

$$r^* = (1 - \beta) \eta L_E^* \quad (38)$$

- Euler equation (unchanged):

$$g^* = g(C) = \frac{(1 - \beta) L_E^* - \rho}{\theta} \quad (39)$$

# Characterizing balanced growth

Solve for the growth rate.

$$\begin{aligned}g(C) &= \frac{(1 - \beta) L_E^* - \rho}{\theta} \\&= g(N) \\&= \eta L_R^* = \eta (1 - L_E^*)\end{aligned}$$

⇒

$$L_E^* = \frac{\theta \eta L + \rho}{(1 - \beta) \eta + \theta \eta} \quad (40)$$

**Scale effects:** larger economies grow faster.

With population growth, output growth explodes.

# Growth without scale effects

- The previous models do not have balanced growth paths when there is population growth.
- The reason is the scale effect:
  - Larger population  $\rightarrow$  more R&D  $\rightarrow$  faster growth.
- Diminishing returns to reproducible factors avoid the scale effect, but also kill endogenous growth.

# Growth without scale effects

- To avoid scale effects, modify the model as follows.
- Innovation:

$$\dot{N}_t = \eta N_t^\phi L_{Rt} \quad (41)$$

$$0 < \phi \leq 1 \quad (42)$$

- Demographics:

$$L_t = e^{nt} \quad (43)$$

$$= L_{Rt} + L_{Et} \quad (44)$$

# Balanced growth

- $g(N) = \eta N_t^{\phi-1} L_{Rt}$ .
- Constant growth requires constant  $N^{\phi-1} L_R$  and

$$g(N) = \frac{n}{1 - \phi} \quad (45)$$

- The growth rate is "**semi-endogenous**:" endogenous, but not responding to changes in agents' choice variables.
- There are still scale effects:
  - Larger economies tend towards higher levels of output per person.

# Avoiding scale effects

- It is possible to write down models that have endogenous growth, but no scale effects (growth does not increase with  $L$ ).
- The idea: Prevent innovator profits from increasing with  $L$ .
- One approach: the number of products increases with  $L$  exactly so that the market size for each variety remains the same.
- Avoiding scale effects requires knife-edge assumptions like this.

- Acemoglu, "Introduction to Modern Economic Growth," ch. 13.
- Romer, "Advanced macro," ch. 3.1-3.4.
- Jones, Charles (2004). "Growth and Ideas."  
<http://elsa.berkeley.edu/~chad/cv.html#Papers>.