

A Model of Growth and Innovation

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- We study a GE model of growth driven by innovation.
- Innovation takes the form of inventing new goods.
- Alternative: Quality ladders.

A Model of Product Innovation

- Demographics: A representative household.
- Preferences:

$$\int_0^{\infty} e^{-\rho t} \frac{C_t^{1-\theta} - 1}{1-\theta} dt \quad (1)$$

- Production of **final goods** from intermediates and labor:

$$Y_t = (1 - \beta)^{-1} \left[\int_0^{N_t} x(v, t)^{1-\beta} dv \right] L^\beta \quad (2)$$

- This is of the Dixit-Stiglitz form: write $[\int x^{1-\beta} dv]^{\frac{1-\beta}{1-\beta}}$ to see that this is a CES aggregator of x .
- Resource constraint:

$$C_t + X_t + Z_t = Y_t$$

- Z : R&D investment.
- X : Inputs into the production of x .

Technology: Intermediate Inputs

- Each unit of x requires ψ units of Y .
- Intermediate inputs fully depreciate in use.

- Investing the final good yields a flow of new patents:

$$\dot{N} = \eta Z_t \quad (3)$$

- Think of this as the aggregate (deterministic) outcome of the (stochastic) innovation efforts of many firms.

Market arrangements

- Final goods and labor markets are competitive.
- The household owns the innovating firms.
- Innovators receive perpetual patents (monopolies) for their varieties.
- Free entry into innovation (ensures zero profits).

Solving Each Agent's Problem

- Maximize period profits by choosing L and $x(v, t)$.
- Normalize the price Y to 1.
- Profits

$$Y_t - w_t L_t - \int_0^{N_t} p^x(v, t) x(v, t) dv \quad (4)$$

where

$$Y_t = (1 - \beta)^{-1} \left[\int_0^{N_t} x(v, t)^{1-\beta} dv \right] L^\beta \quad (5)$$

- Demand function (cf. the Dixit Stiglitz discussion):

$$x(v, t) = L p^x(v, t)^{-1/\beta} \quad (6)$$

- **Labor demand** is standard:

$$w_t = \beta Y_t / L_t \quad (7)$$

- **Solution** to the firm's problem: $L_t, x(v, t)$ that satisfy the "2" first-order conditions.

Intermediate input producers

- Problem after inventing a variety.
- x is produced at constant marginal cost ψ .
- Maximize present value of profits

$$V(v, t) = \int_t^{\infty} e^{-rs} \pi(v, s) ds \quad (8)$$

- Instantaneous profits are

$$\pi(v, t) = (p^x(v, t) - \psi) x(v, t) \quad (9)$$

- With constant demand elasticity:

$$p^x = \psi / (1 - \beta) \quad (10)$$

- Profits are

$$\pi(v, t) = \psi \frac{\beta}{1 - \beta} x(v, t) \quad (11)$$

- Solution: A constant p^x .

- The household holds shares of all intermediate input firms.
- Each firm produces a stream of profits.
- New firms issue new shares.
- But: the details don't matter to the household.
- There simply is an asset with rate of return r .
- Euler equation is standard:

$$g(C) = \frac{r - \rho}{\theta} \quad (12)$$

- Invoke Walras' law - so you never have to write down the budget constraint!

- Objects: $C_t, X_t, Z_t, x(v, t), V(v, t), N_t$ and prices $p^x(v, t), r(t), w(t)$.
- Conditions:
 - "Everybody maximizes." (see above)
 - Markets clear.
 - 1 Goods: resource constraint.
 - 2 Shares: omitted b/c I did not write out the household budget constraint.
 - 3 Intermediates: implicit in notation.
 - Innovation effort satisfies a **free entry** condition: present value of profits equals 0.

Symmetric Equilibrium

- Assume that all varieties v share the same x , V and p^x .

Equilibrium: Characterization

- The growth rate follows from the Euler equation: $g(C) = (r - \rho)/\theta$.
 - We need to solve for r .
- **Simplifications:**
 - Normalize $\psi = 1 - \beta$ so that $p^x = 1$.
 - Why can I do that?
 - Focus on balanced growth paths.

Equilibrium: Characterization

- Free entry will determine the interest rate
- Spend 1 to obtain η new patents, each valued at $V(t)$

$$\eta V(t) = 1 \quad (13)$$

- Assume there is innovation.
 - Then V is constant over time.
- With balanced growth and constant profits (to be shown):

$$V = \pi/r \quad (14)$$

$$\pi(t) = \psi \frac{\beta}{1-\beta} x(t) \quad (15)$$


$$= \beta x(t) \quad (16)$$

- Demand for intermediates:

$$\begin{aligned} x(t) &= L p^x(t)^{-1/\beta} \\ &= L \end{aligned}$$

- Profits: $\pi = \beta L$.
- Free entry:

$$\eta \beta L / r = 1 \quad (17)$$

- This is the closed form solution for r .
- Balanced **growth** rate then follows from the Euler equation. 

Equilibrium: Characterization

- Production function for final goods:

$$Y = \frac{N_t L}{1 - \beta} \quad (18)$$

- Labor demand (from firm's FOC):

$$w_t = \beta \frac{Y_t}{L_t} = \frac{\beta}{1 - \beta} N_t \quad (19)$$

- Total expenditure on intermediates:

$$X_t = N_t p^x x_t = N_t L \quad (20)$$

(the book has an additional $1 - \beta$ term).

$$g(C) = \frac{\eta\beta L - \rho}{\theta}$$

- Larger economies (L) grow faster.
- Mechanical reason:
 - Innovation technology is linear in goods.
 - Larger economy \rightarrow higher $Y \rightarrow$ higher $Z \rightarrow$ faster growth.

No Transition Dynamics

- Period profits π are constant at βL .
- At any moment we need $\eta V = 1$.
- Constant V is only possible with constant r .
- Intuition: There is a reduced form AK structure.

$$Y_t = \frac{L}{1 - \beta} N_t$$
$$\dot{N}_t = \eta s_z Y_t$$

Pareto Efficient Allocation

Two distortions prevent efficiency of equilibrium:

- 1 Monopoly pricing.
- 2 Inefficient innovation due to aggregate demand externality.

Planner's Problem

Solve in two stages:

- 1 Given N , find optimal static allocation $x(v, t)$.
 - That is: maximize $Y - X$ which is available for consumption and investment.
 - An odd feature of the model: goods are produced from goods without delay.
- 2 Given the reduced form production function from #1, find optimal Z .

Static Allocation

Given N , choose $x(v, t)$ to maximize $Y - X$:

$$\max (1 - \beta)^{-1} L^\beta \int_0^{N_t} x(v, t)^{1-\beta} dv - \int_0^{N_t} \psi x(v, t) dv \quad (21)$$

First-order condition

$$L^\beta x^{-\beta} = \psi \quad (22)$$

with $\psi = 1 - \beta$:

$$x = (1 - \beta)^{-1/\beta} L \quad (23)$$

- Next: find $Y - X$.
- $X = NX = (1 - \beta)^{-1/\beta} L$.
- Reduced form production function:

$$Y_t = (1 - \beta)^{-1} L^\beta N [(1 - \beta)^{-1/\beta} L]^{1-\beta} \quad (24)$$

$$= (1 - \beta)^{-1/\beta} L N_t \quad (25)$$

- Net output

$$\begin{aligned} Y - X &= (1 - \beta)^{-1/\beta} L N - (1 - \beta) (1 - \beta)^{-1/\beta} L N \\ &= (1 - \beta)^{-1/\beta} \beta L N \end{aligned} \quad (26)$$

$$\max \int_0^{\infty} e^{-\rho t} \frac{C_t^{1-\theta} - 1}{1-\theta} dt$$

subject to

$$\dot{N} = \eta Z$$

$$Y = (1-\beta)^{-1/\beta} \beta L N = C + Z$$

Or

$$\dot{N} = A N - \eta C \tag{27}$$

$$A = \eta (1-\beta)^{-1/\beta} \beta L \tag{28}$$

$$H = \frac{C^{1-\theta} - 1}{1-\theta} + \mu [AN - \eta C] \quad (29)$$

FOC

$$\partial H / \partial C = C^{-\theta} - \mu \eta = 0 \quad (30)$$

$$\partial H / \partial N = \rho \mu - \dot{\mu} = \mu A \quad (31)$$

The same as in an AK model with

$$A = \eta (1 - \beta)^{-1/\beta} \beta L \quad (32)$$

we have

$$\dot{C}/C = \frac{A - \rho}{\theta} \quad (33)$$

Comparison with CE

- CE interest rate: $\eta\beta L$.
- Planner's "interest rate:" $(1 - \beta)^{-1/\beta} \eta\beta L$.
- The planner chooses faster growth.
- Intuition:
 - CE under-utilizes the fruits of innovation: x is too low.
 - This reduces the value of innovation.

- One might be tempted to reduce monopoly power.
- A policy that encourages competition (e.g. less patent protection, forcing lower p^x) reduces the static price distortion.
- But it also reduces growth: innovation is less valuable.
- Similar result for shorter patents.
- Policy trades off static efficiency and incentives for innovation.

Knowledge Spillovers

Ideas Produced From Labor

- The previous model had endogenous growth because ideas were produced from a **reproducible factor**: goods.
- If ideas are produced from (non-reproducible) labor: there is no sustained growth.
- As we showed before: sustained growth requires constant returns to reproducible factors.
 - More precisely: returns to reproducible factors must be **asymptotically bounded from below**.

Example

Assume $\dot{N}_t = \eta Z_t^\alpha L_{Rt}^{1-\alpha}$. Show that the balanced growth rate is 0 unless $\alpha = 1$.

Knowledge Spillovers

- If any non-reproducible factors are used in innovation, knowledge spillovers are required to sustain growth.
- Knowledge spillovers means: N appears in the innovation production function.
- Economic idea: "standing on the shoulders of giants"
- Problem: A **knife-edge** parameter assumption is needed for endogenous growth.
 - Some parameters must sum to 1.
 - This is always true because we need **constant returns to reproducible factors**.

Knowledge spillover model

- Keep everything the same, except the production of ideas:

$$\dot{N}_t = \eta N_t L_{Rt} \quad (34)$$

- We show later: linearity in N is required for endogenous growth.
- Labor now has 2 uses:
 - produce goods: L_E
 - produce ideas: L_R

- Resource constraint:

$$L = L_{Rt} + L_{Et} \quad (35)$$

- Note: this does not change the problems of household, final goods firms, or intermediate input firms.

- Euler equation is still: $g(C) = (r - \rho)/\theta$.
- Interest rate is determined by free entry: $V = \pi/r$.
- But now the cost of creating a new patent is different:

$$\eta N_t V_t = w_t \quad (36)$$

- the same as before but R&D productivity is now ηN instead of η .

Balanced growth rate

- Wage rate (unchanged):

$$w_t = \frac{\beta}{1 - \beta} N_t \quad (37)$$

- Profits earned by monopolists (unchanged):

$$\pi_t = \beta L_{Et} \quad (38)$$

- Sub wage rate into free entry:

$$\eta N_t \frac{\beta L_E}{r} = \frac{\beta}{1 - \beta} N_t \quad (39)$$

\implies

$$r^* = (1 - \beta) \eta L_E^* \quad (40)$$

- Euler equation (unchanged):

$$g^* = g(C) = \frac{(1 - \beta)L_E^* - \rho}{\theta} \quad (41)$$

- Almost done - just need to find L_E .
- Balanced growth requires

$$g(C) = g(N) = \eta L_R^* = \eta (1 - L_E^*) \quad (42)$$

Solve for the growth rate.

$$g(C) = \frac{(1 - \beta)L_E^* - \rho}{\theta \eta (1 - L_E^*)}$$



$$L_E^* = \frac{\theta \eta L + \rho}{(1 - \beta) \eta + \theta \eta} \quad (43)$$

Scale effects: larger economies grow faster.

With population growth, output growth explodes.

Growth without scale effects

- The previous models do not have balanced growth paths when there is population growth.
- The reason is the scale effect:
 - Larger population \rightarrow more R&D \rightarrow faster growth.
- Diminishing returns to reproducible factors avoid the scale effect, but also kill endogenous growth.

Growth without scale effects

- To avoid scale effects, modify the model as follows.
- Innovation:

$$\dot{N}_t = \eta N_t^\phi L_{Rt} \quad (44)$$

$$0 < \phi \leq 1 \quad (45)$$

- Demographics:

$$L_t = e^{nt} \quad (46)$$

$$= L_{Rt} + L_{Et} \quad (47)$$

Balanced growth

- $g(N) = \eta N_t^{\phi-1} L_{Rt}$.
- Constant growth requires constant $N^{\phi-1} L_R$ and

$$g(N) = \frac{n}{1 - \phi} \quad (48)$$

- The growth rate is "**semi-endogenous**:" endogenous, but not responding to changes in agents' choice variables.
- There are still scale effects:
 - Larger economies tend towards higher levels of output per person.

Avoiding scale effects

- It is possible to write down models that have endogenous growth, but no scale effects (growth does not increase with L).
- The idea: Prevent innovator profits from increasing with L .
- One approach: the number of products increases with L exactly so that the market size for each variety remains the same.
- Avoiding scale effects requires knife-edge assumptions like this.

- Acemoglu, "Introduction to Modern Economic Growth," ch. 13.
- Romer, "Advanced macro," ch. 3.1-3.4.
- Jones, Charles (2004). "Growth and Ideas."
<http://elsa.berkeley.edu/~chad/cv.html#Papers>.