

1 Technology Adoption

[Based on Romer (2001) question 3.12] The world consists of 2 regions: $i = N, S$. Output is produced according to

$$Y_i = K_i^\alpha (A_i [1 - a_{Li}] L_i)^{1-\alpha}$$

Capital is accumulated according to

$$\dot{K}_i = s_i Y_i$$

New technologies are developed in the North:

$$\dot{A}_N = B a_{LN} L_N A_N.$$

The South learns from the North:

$$\dot{A}_S = \mu a_{LS} L_S (A_N - A_S)$$

as long as $A_N > A_S$. Otherwise $\dot{A}_S = 0$.

Labor supplies (L_i) are constant over time.

1. Determine the long-run growth rate of Y_N .
2. Define $Z = A_S/A_N$. Derive the law of motion for Z . Is Z stable? If so, which value does it converge to? What is the long-run growth rate of Y_S ?
3. Assume $a_{LN} = a_{LS}$ and $s_N = s_S$. Find the balanced growth output ratio Y_N/Y_S .

1.1 Answer: Technology adoption

1. The growth rates obey

$$g(Y_i) = \alpha g(K_i) + (1 - \alpha) (g(A_i) + n) \tag{1}$$

$$g(K_i) = s_i Y_i / K_i \tag{2}$$

$$g(A_N) = B a_{LN} L_N \tag{3}$$

$$g(A_S) = \mu a_{LS} L_S (A_N/A_S - 1) \tag{4}$$

In the long-run

$$g(Y_i) = g(K_i)$$

and therefore

$$g(Y_i) = g(A_i) + n \tag{5}$$

2. The law of motion for Z is

$$\begin{aligned} g(Z) &= g(A_S) - g(A_N) \\ &= \mu a_{LS} L_S (1/Z - 1) - B a_{LN} L_N \end{aligned}$$

This has a unique, stable steady state. Its value is

$$1/Z^* = 1 + \frac{B a_{LN} L_N}{\mu a_{LS} L_S}$$

This implies that larger adopting countries have higher productivity (Z) - the scale effect in action.

3. Equals s_i implies equal K/Y . Write the production function as

$$Y_i/L_i = (K_i/Y_i)^{\alpha/(1-\alpha)} A_i (1 - a_{Li}) \tag{6}$$

Then the ratio of output per worker is

$$y_S/y_N = Z \frac{1 - a_{LS}}{1 - a_{LN}} \tag{7}$$

so that larger adopting countries are richer.

2 Delayed Adoption

[Based on Romer (2001) question 3.13] The world consists of two regions: $i = N, S$. The North is described by

$$\begin{aligned} Y_N &= A_N (1 - a_{LN}) L_N \\ \dot{A}_N &= a_L L_N A_N \end{aligned}$$

The South uses technologies developed in the North with a lag of τ :

$$\begin{aligned} A_S(t) &= A_N(t - \tau) \\ Y_S(t) &= A_S(t) L_S \end{aligned}$$

Labor inputs are constant over time.

If $g(Y) = 0.03$, how old must technologies be in the South to yield a 10-fold output gap? Assume a_L is close to 0 and $n = 0.01$.

2.1 Answer: Delayed adoption

From the production functions

$$\frac{y_N(t)}{y_S(t)} = \frac{(1 - a_{LN}) A_N(t)}{A_N(t - \tau)}$$

We next find the ratio $A_N(t)/A_N(t - \tau)$ which measures how far behind the South is in terms of technology. A grows at rate

$$g(A) = g(Y) - n$$

so that

$$A_N(t)/A_N(t - \tau) = e^{[g(Y) - n]\tau}.$$

If $a_{LN} = 0$ we need the A gap to equal 10:

$$10 = e^{0.02\tau}$$

or

$$\begin{aligned}\tau &= \ln(10)/0.02 \\ &\simeq 115\end{aligned}$$

In words: the technology gap would have to be 115 years. It follows that delayed adoption by itself cannot account for large cross-country income gaps. It must also be the case that technologies are used inefficiently.

3 Ideas produced from capital and labor

Consider the following version of a Romer model. The Solow block:

$$Y = K_Y^\alpha L_Y^{1-\alpha} \tag{8}$$

$$\dot{K} = A s_K Y - \delta K \tag{9}$$

$$L_t = L_0 e^{nt} \tag{10}$$

Production of ideas:

$$\dot{A} = \delta K_A^\beta L_A^\lambda A^\phi \tag{11}$$

$$0 < \phi < 1 \tag{12}$$

Constant behavior:

$$L_Y = s_Y L \quad (13)$$

$$L_A = s_A L \quad (14)$$

$$K_A = a K \quad (15)$$

$$K_Y = (1 - a) K \quad (16)$$

1. Find the balanced growth rates of A, K, Y .
2. Derive expressions that give the *growth rates* of $g(A)$ and $g(K/L)$ as functions of $g(A), g(K/L)$, and parameters. Set $\delta = 0$.
3. Graph curves representing constant $g(A)$ and $g(K/L)$. Place $g(A)$ on the x-axis and $g(K/L)$ on the y-axis.
4. Which assumptions on the parameters do you need for a balanced growth path to exist (the two curves to intersect)?
5. Is the balanced growth path stable in the case where it exists?

3.1 Answer: Ideas produced from capital and labor

1. Start from

$$\begin{aligned} (1 - \phi) g(A) &= \lambda n + \beta g(K/L) + \beta n \\ g(A) &= (1 - \alpha) g(K/L) \\ g(Y/L) &= g(K/L) \end{aligned}$$

These are three equations which can be solved for the 3 growth rates. Some algebra yields

$$g(A) = n \frac{\lambda + \beta}{1 - \phi - \beta / (1 - \alpha)} \quad (17)$$

- 2.

$$\begin{aligned} g(g(K)) &= g(A) - (1 - \alpha) g(K/L) \\ g(g(A)) &= (\beta + \lambda) n + \beta g(K/L) - (1 - \phi) g(A) \end{aligned}$$

3.-4. These are straight lines with slopes $(1 - \phi) / \beta$ and $1 / (1 - \alpha)$. For these to intersect, we need the denominator in (17) to be positive.

5. Draw the phase diagram to see that the BGP is stable.

4 Learning by doing

Consider the following model. Output (Y) is produced from capital (K) and labor (L):

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha} \quad (18)$$

with $0 < \alpha < 1$. Labor grows at the constant rate n :

$$L_t = e^{nt} \quad (19)$$

Capital is accumulated according to

$$\dot{K}_t = sY_t. \quad (20)$$

The saving rate s is constant. Knowledge is accumulated as a by-product of production:

$$\dot{A}_t = BY_t^\varphi$$

with $0 < \varphi < 1$ and $B > 0$.

1. Derive the *balanced* growth rates of Y , K , and A .
2. Does a change in s affect long-run growth? Explain the intuition. (You can do this even if you did not answer 1.)
3. Does a change in n affect long-run growth? Explain the intuition. (You can do this even if you did not answer 1.)
4. Derive an expression for the balanced growth *level* of $Y(t)$ as a function of $L(t)$ and parameters. (If you could not solve for the balanced growth rate in 1, simply call it g here.)
5. Now assume that $\varphi = 1$ and $n = 0$. Solve for the balanced growth rate of Y as a function of s , L , and parameters. Explain why a larger L implies faster balanced growth. Hint: Find expressions for ratios that are constant on the balanced growth path ($Y/K, Y/A, g_Y$).

4.1 Answer: Learning by doing

1. Balanced growth rates. First, write all model equations in growth rates:

$$g_Y = \alpha g_K + (1 - \alpha)(n + g_A) \quad (21)$$

$$g_K = sY/K \quad (22)$$

$$g_A = BY^\varphi/A \quad (23)$$

From (22) we find that $g_Y = g_K$ on the BGP. Then from the production function $g_Y = n + g_A$. Therefore:

$$g_Y = g_K = g_A + n = \frac{n}{1 - \varphi} \quad (24)$$

$$g_A = \frac{\varphi n}{1 - \varphi} \quad (25)$$

2. s does not affect long-run growth. A one-time increase in s raises K and therefore Y . g_A rises temporarily, but eventually diminishing returns in the production of ideas cause growth to slow down again.

3. n does affect long-run growth. The reason is the same as in the R&D model. Faster output growth implies faster growth in inputs to innovation and hence faster growth of ideas.

4. Balanced growth output The strategy for finding levels is: Start from the production function. We need to substitute out the endogenous K and A to express Y as a function of exogenous objects only. Use the laws of motions to express K and A as functions of Y and other exogenous objects. Substitute these into the production function. Solve for Y .

The BGP capital-output ratio is

$$g_K = sY/K. \quad (26)$$

From this we find

$$K = sY/g_K. \quad (27)$$

From the law of motion for A :

$$A = B Y^\varphi/g_A. \quad (28)$$

Substitute into the production function:

$$Y = \left(\frac{sY}{g_Y}\right)^\alpha L^{1-\alpha} \left(\frac{B Y^\varphi}{g_A}\right)^{1-\alpha} \quad (29)$$

or

$$Y^{(1-\alpha)(1-\varphi)} = \left(\frac{s}{g_Y}\right)^\alpha L^{1-\alpha} \left(\frac{B}{g_A}\right)^{1-\alpha} \quad (30)$$

$$= s^\alpha L^{1-\alpha} (B/\varphi)^{1-\alpha} \frac{1-\varphi}{n} \quad (31)$$

5. Balanced growth rate. Now the balanced growth rates satisfy:

$$g = g_Y = g_K = g_A = BY/A = sY/K \quad (32)$$

Start from the law of motion for A :

$$\begin{aligned} g &= BY/A \\ &= B (K/A)^\alpha L^{1-\alpha} \end{aligned}$$

We need to find K/A , then we are done. From

$$BY/A = g = sY/K$$

we have

$$K/A = s/B$$

and therefore

$$g = (BL)^{1-\alpha} s^\alpha \quad (33)$$

Larger economies (higher L) grow faster. This is the usual scale effect. Larger economies devote more resources to innovation.

5 Endogenous growth without scale effects

There is one final good which is produced by combining intermediates according to the production function

$$C = \left(\int_0^B Y_i^{1/\theta} di\right)^\theta \quad (34)$$

We call final output C because it can only be consumed. $\theta > 0$.

There are B intermediate goods (Y_i). Each is produced from labor $L_{Y,i}$ and ideas M_i according to

$$Y_i = M_i^\sigma L_{Y,i} \quad (35)$$

Innovation must be undertaken for each variety separately:

$$\dot{M}_i = \delta M_i L_{M,i} \quad (36)$$

Finally, for reasons that we will see below, assume that $B = L^\beta$.

The labor constraint is now

$$L = \int_0^B (L_{Y,i} + L_{M,i}) di \quad (37)$$

We shall again assume that the labor allocation is constant such that fraction s of the total labor endowment is allocated to innovation

$$sL = \int_0^B L_{M,i} di \quad (38)$$

Assume symmetry: all varieties are identical.

1. Show that

$$Y_i = M^\sigma (1 - s) L^{1-\beta} \quad (39)$$

for all varieties i . Let's call this output level Y .

2. Derive an expression for the growth rate of per capita output $c = C/L$.
3. Discuss the effects of policies which change s on long-run growth. How does your answer depend on the value of β . Can you provide intuition for your finding?

Note: If you have trouble with the integrals, simply replace them with sums. The result is the same.

5.1 Answer: Endogenous growth without scale effects

1. With symmetry the model equations become

$$\begin{aligned} M_i &= M \\ L_{M,i} &= sL/B = sL^{1-\beta} \\ L_{Y,i} &= (1-s)L/B = (1-s)L^{1-\beta} \end{aligned}$$

Substitute into the production function for varieties:

$$Y_i = Y = M^\sigma L_{Y,i} = M^\sigma (1-s) L^{1-\beta} \quad (40)$$

2. Growth rate: With symmetry:

$$\begin{aligned} C &= B^\theta Y \\ c &= C/L = B^\theta M^\sigma (1-s)L^{-\beta} \end{aligned} \tag{41}$$

Taking growth rates

$$\begin{aligned} g(c) &= \theta g(B) + \sigma g(M) - \beta n \\ g(M) &= \delta s L^{1-\beta} \\ g(B) &= \beta n \end{aligned}$$

Thus

$$g(c) = (\theta - 1)\beta n + \sigma \delta s L^{1-\beta} \tag{42}$$

3. Long-run effects of policies If $\beta > 1$ the model behaves much like the standard Romer model in that policies (changing s) do not affect long-run growth. The reason is that $L^{1-\beta} \rightarrow 0$ over time. However, if $\beta = 1$, the $L^{1-\beta}$ term does not go to 0 as L becomes large. Then changes in s affect long-run growth. If $\beta < 1$, then the model only has a balanced growth path for the case of $n = 0$. Otherwise it has exploding growth.

What is the intuition for the absence of scale effects? The scale effect usually arises because a larger market makes R&D more productive or more profitable. In this simple model, profitability is not an issue because s is exogenous. Here, the scale effect comes from having more researchers working on each variety if there is a lot of labor around. That is

$$L_{M,i} = sL/B = sL^{1-\beta}$$

is high unless $\beta = 1$. But if $\beta = 1$, the number of varieties rises exactly in proportion to L , so that $L_{M,i}$ remains unchanged as L grows.

How can one motivate the assumption that B is proportional to L . Young (1998) shows that this can arise as an equilibrium outcome, if households have a preference for variety. With a larger population, per capita output rises. Consumers could increase how much they consume of each variety. Then the scale effect would be back. But with the right preferences they will hold Y_i constant and increase the number of varieties instead – the scale effect is gone. However, in setting $\beta = 1$ another knife-edge assumption is introduced.