

# Endogenous Growth: AK Model

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- Why do countries grow?
  - A question with large welfare consequences.
- We need models where growth is endogenous.
- The simplest model is a variation of the Ramsey model.
  - Growth can be sustained if the MPK is bounded below.
  - AK model

# Necessary Conditions for Sustained Growth

- How can growth be sustained without exogenous productivity growth?
- A necessary condition: **constant returns to the reproducible factors**.
  - Vaguely: The production function must be linear in factors that can be produced.
- This motivates a simple class of models in which
  - 1 only  $K$  can be produced and
  - 2 the production function is  $AK$ .
- This can be thought of as a reduced form for more complex models (we'll see examples).

# Solow AK model

- To see what is required for endogenous growth, consider the Solow model:

$$g(k) = sf(k)/k - (n + \delta) \quad (1)$$

- Positive long-run growth requires: As  $k \rightarrow \infty$  it is the case that

$$f(k)/k > n + \delta \quad (2)$$

- L'Hopital's rule implies (if  $f'$  has a limit):

$$\lim f(k)/k = \lim f'(k) \quad (3)$$

- Sustained growth therefore requires:

$$\lim_{k \rightarrow \infty} f'(k) > n + \delta \quad (4)$$

# Necessary Conditions for Sustained Growth

- This argument is more general than the Solow model.
  - It does not matter how  $s$  is determined.
- If  $\lim_{k \rightarrow \infty} f'(k)$  exists, the production function has **asymptotic constant returns to scale**.

$$f(k) \rightarrow Ak + B \quad (5)$$

- It is fine to have diminishing returns for finite  $k$ .

①  $f(k) = Ak + Bk^\alpha$  with  $0 < \alpha < 1$

①  $f(k)/k \rightarrow A$

② CES production function with high elasticity of substitution:

$$F(K, L) = [\mu K^\theta + (1 - \mu)L^\theta]^{1/\theta} \quad (6)$$

①  $f(k) = [\mu k^\theta + 1 - \mu]^{1/\theta}$

② Elasticity of substitution:  $\varepsilon = (1 - \theta)^{-1}$ .

③ If  $\theta > 0$  [ $\varepsilon > 1$ ],  $f(k)/k \rightarrow \mu$ .

- In the Solow model, assume  $f(k) = Ak$ .
- Law of motion:

$$g(k) = sA - n - \delta \quad (7)$$

- Changes in parameters alter the growth rate of  $k$ .
- The model does not have any transitional dynamics:  $k$  always grows at rate  $sA - n - \delta$ .

# AK Solow Model

- It is not necessary to have constant returns in all sectors of the economy.
- Imagine that  $c$  is produced from  $k$  with diminishing returns to scale:  $c = [(1 - s)Ak]^\varphi$  with  $\varphi < 1$ .
- The law of motion for  $k$  is unchanged (so is the balanced growth rate of  $k$ ).
- This model still has a balanced growth path with a strictly positive growth rate, but not  $c$  and  $k$  grow at different constant rates:

$$g(c) = \varphi g(k) \tag{8}$$

# AK Neoclassical Growth Model

# AK neoclassical growth model

- This model adds optimizing consumers to the  $Ak$  model.
- Households maximize

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t) dt \quad (9)$$

subject to the flow budget constraint

$$\dot{k} = (r - n)k - c \quad (10)$$

- There is no labor income because in the  $Ak$  world all income goes to capital.

# AK neoclassical growth model

- For balanced growth we need

$$u(c) = c^{1-\sigma}/(1-\sigma) \quad (11)$$

- The optimality conditions are the same as in the Cass-Koopmans model:

$$g(c) = (r - \rho)/\sigma$$

and the transversality condition (assuming constant  $r$ )

$$\lim_{t \rightarrow \infty} k_t e^{-(r-n)t} = 0 \quad (12)$$

- Firms maximize period profits.
- The first-order condition is  $r = A - \delta$ .

- An allocation:  $c(t), k(t)$ .
- A price system:  $r(t)$ .
- These satisfy:
  - ① Household: Euler, budget constraint (TVC).
  - ② Firm: 1 foc.
  - ③ Market clearing:

$$\dot{k} = Ak - (n + \delta)k - c \quad (13)$$

- Simplify into a pair of differential equations:

$$\dot{k} = (A - \delta - n)k - c \quad (14)$$

$$g(c) = (A - \delta - \rho)/\sigma \quad (15)$$

- Boundary conditions:  $k_0$  given and the TVC.

- We need restrictions on the parameters that ensure bounded utility.
- Lifetime utility is

$$\int_0^{\infty} e^{-(\rho-n)t} [c_0 e^{g(c)t}]^{1-\sigma} dt / (1-\sigma) \quad (16)$$

- Boundedness then requires that  $n - \rho + (1 - \sigma)g(c) < 0$ .
- Instantaneous utility cannot grow faster than the discount factor  $(\rho - n)$ .

- This model has no transitional dynamics.
- Consumption growth is obviously constant over time.
- To show that  $g(k)$  is constant: we need to solve for  $k(t)$  in closed form.
- This is boring math and we skip over it...

## Digression: Solving for $k(t)$ I

- Law of motion:

$$\dot{k}_t = (A - \delta - n) k_t - c_0 \exp\left(\frac{A - \delta - \rho}{\sigma} t\right) \quad (17)$$

- Solution to  $\dot{x} = ax - b(t)$  is

$$x_t = x_0 e^{at} - e^{at} \int_0^t e^{-as} b(s) ds \quad (18)$$

- To verify:

$$\dot{x}_t = ax_0 e^{at} - a e^{at} \int_0^t e^{-as} b(s) ds - e^{at} e^{-at} b(t) \quad (19)$$

$$= ax_t - b(t) \quad (20)$$

## Digression: Solving for $k(t)$ II

- Define

$$a = r - n = A - \delta - n > 0 \quad (21)$$

$$b = g_c = \frac{A - \delta - \rho}{\sigma} > 0 \quad (22)$$

- Then

$$k_t = k_0 \exp(at) - \exp(at) \int_0^t c_0 \exp([-a + b]s) ds \quad (23)$$

- Note:

$$\int_0^t e^{zs} ds = \frac{e^{zt} - 1}{z} \quad (24)$$

## Digression: Solving for $k(t)$ III

- Therefore:

$$k_t = k_0 e^{at} - \frac{c_0}{b-a} e^{at} \left[ e^{(b-a)t} - 1 \right] \quad (25)$$

$$= \left[ k_0 + \frac{c_0}{b-a} \right] e^{at} - \frac{c_0}{b-a} e^{bt} \quad (26)$$

- Now we show that  $g(k)$  is constant:  $k_t = k_0 e^{bt}$ .
- Transversality:

$$\lim_{t \rightarrow \infty} e^{(r-n)t} k_t = 0 \quad (27)$$

- Note that  $a = r - n = A - \rho - n$ .
- If  $b > a$ :  $g(k) \rightarrow b > a$  and TVC is violated.
- So we need  $b < a$ .

## Digression: Solving for $k(t)$ IV

- With  $b < a$  capital grows at rate  $a$ , unless the term in brackets is 0:

$$k_0 - \frac{c_0}{b-a} = 0 \quad (28)$$

- If  $g(k) = a$ , then  $g(e^{-(r-n)t}k_t) = 0$  - because  $a = r - n$ .
  - That violates TVC.
- The only value of  $c_0$  consistent with TVC is the one that sets the term in brackets to 0.
- It implies that  $k$  always grows at rate  $b$ .

- We can solve for  $c/k$  and the saving rate.

$$\begin{aligned}g(k) - g(c) &= [A - \delta - n - c/k] - (A - \delta - \rho)/\sigma = 0 \\c/k &= A - \delta - n - (A - \delta - \rho)/\sigma\end{aligned}$$

- And the gross savings rate is

$$\begin{aligned}s &= (\dot{K} + \delta K)/AK \\&= [g(K) + \delta]/A \\&= [g(c) + n + \delta]/A \\&= [(A - \delta - \rho)/\sigma + n + \delta]/A\end{aligned}$$

- The savings rate is high, if ( $\sigma$ ,  $\rho$  or  $A$ ) are low, or if  $n$  is high.

The AK model has a very simple equilibrium.

- 1 The saving rate is constant.
- 2 All growth rates are constant.

This is very convenient, but also very limiting in many applications.

# How to think about AK models?

- In the data, there is at least one non-reproducible factor: labor.
- Do models with constant returns to reproducible factors make sense?
- The best way of thinking about AK models:
  - a reduced form for a model with multiple factors
  - there may be transition dynamics, but it does not matter if you are interested in long-run issues
  - there may be fixed factors, but it does not matter if there are constant returns to reproducible factors.

# Examples: AK as reduced form

- 1 Human capital:  $F(K, hL)$  with  $K$  and  $h$  reproducible.
- 2 Externalities:
  - 1 Romer (1986). For the firm  $F(k_i, l_i; K) = K^{1-\alpha} k_i^\alpha l_i^\theta$
  - 2 Firms take  $K$  as given - diminishing returns to  $k_i$ .
  - 3 In equilibrium:  $K = \sum k_i$  - constant returns to scale to  $K$ .
- 3 Increasing returns to scale at the firm level:  $y = Ak^\alpha l^{1-\alpha}$ 
  - 1  $A$  can be produced somehow - R&D.
  - 2 Need imperfect competition.

## Example: Lucas (1988)

- A classic endogenous growth paper.
- Growth is due to human capital accumulation.
- The model has an AK reduced form.

# Model: Lucas (1988)

- Demographics:
  - A representative, infinitely lived household.
- Preferences:

$$\int_0^{\infty} e^{-\rho t} u(c_t) dt \quad (29)$$

$$u(c) = c^{1-\sigma} / (1-\sigma) \quad (30)$$

- Technology:

$$\dot{k} + c = k^{\alpha} (uh)^{1-\alpha} - \delta k \quad (31)$$

$$\dot{h} = B(1-u)h - \delta h \quad (32)$$

- Law of motion for  $h$ :

$$g(h) = B(1 - u) - \delta \quad (33)$$

- Law of motion for  $k$ :

$$g(k) + c/k = (uh/k)^{1-\alpha} - \delta \quad (34)$$

- Therefore:

$$g(c) = g(k) = g(h) \quad (35)$$

- Current value Hamiltonian:

$$H = u(c) + \lambda [B(1 - u)h - \delta] + \mu [f(k, h, u) - \delta k - c] \quad (36)$$

- FOCs:

$$u'(c) = \mu \quad (37)$$

$$\lambda Bh = \mu f_u \quad (38)$$

$$\rho\lambda - \dot{\lambda} = \lambda [B(1 - u) - \delta] + \mu f_h \quad (39)$$

$$\rho\mu - \dot{\mu} = \mu [f_k - \delta] \quad (40)$$

- Euler equation

$$g(c) = \frac{f_k - \delta - \rho}{\sigma} \quad (41)$$

- From FOC for  $h$ :

$$-g(\lambda) = B(1-u) - \delta - \rho + \mu/\lambda f_h \quad (42)$$

- FOC for  $u$ :

$$\frac{\mu}{\lambda} f_h = \frac{B h f_h}{f_u} = B u \quad (43)$$

- Substitute into FOC for  $h$ :

$$-g(\lambda) = B - \delta - \rho \quad (44)$$

- This is an exogenous constant!

# Balanced growth

- Constant  $f_k$  requires constant  $k/h$ .
- Then

$$\frac{\mu}{\lambda} = \frac{Bh}{f_u} \quad (45)$$

requires constant  $\mu/\lambda$ .

- Then  $g(u_c) = -g(\mu) = -g(\lambda) = B - \delta - \rho$ .
- This determines the interest rate:

$$r = f_k - \delta = B - \delta \quad (46)$$

- The balanced growth rate is determined by the linear human capital technology:

$$g(c) = \frac{B - \delta - \rho}{\sigma} \quad (47)$$

- The household has 2 assets:  $k$  and  $h$ .
- One asset has a constant rate of return:
  - give up 1 unit of time to gain a fixed increment of future income
  - regardless of current values of  $k$  and  $h$ .
- This pins down the interest rate on the other asset by no arbitrage.

# Phase Diagram With Endogenous Growth

# Phase Diagram with Endogenous Growth

- We study an endogenous growth model with transitional dynamics.
- As an example of a phase diagram with endogenous growth.

- We modify the  $Ak$  model's production function:

$$H(K, L) = AK + F(K, L) \quad (48)$$

- In intensive form

$$h(k) = Ak + f(k)$$

where  $F(K, L) = Lf(k)$  satisfies Inada conditions and has constant returns to scale in  $K$  and  $L$  jointly.

- For simplicity, assume  $f(k) = k^\alpha$  with  $\alpha < 1$ .

- The only change to the equilibrium conditions of the  $Ak$  model: the marginal product of capital is not  $A$  but

$$H_K(K, L) = A + F_K(K, L) = A + f'(k) \quad (49)$$

- Laws of motion:

$$\dot{k} = (A + f(k)/k - \delta - n)k - c \quad (50)$$

$$g(c) = (A + f'(k) - \delta - \rho)/\sigma \quad (51)$$

- Asymptotically,  $f'(k) \rightarrow 0$  and the model becomes  $Ak$ .

# Phase Diagram with Endogenous Growth

- How to draw a phase diagram when  $c$  and  $k$  grow at endogenous rates?
- Answer: Find ratios that are constant asymptotically:  $z = h(k)/k$  and  $\chi = c/k$ .
- Equivalently, **detrend** the model and then draw the phase diagram.

$$g(z) = g(h(k)) - g(k) \quad (52)$$

$$g(\chi) = g(c) - g(k) \quad (53)$$

- We therefore need to find expressions for  $g(h(k))$ ,  $g(k)$ , and  $g(c)$  in terms of  $z$  and  $\chi$  only.
- First rewrite (50) as

$$g(k) = z - \delta - n - \chi \quad (54)$$

- Next,  $g(c) = [h'(k) - \delta - \rho] / \sigma$ .
  - We need to replace  $h'(k)$ .
- Note that

$$h'(k) = A + \alpha f(k)/k = A + \alpha (z - A) = \alpha z + (1 - \alpha) A$$

- Use this to rewrite (51) as

$$g(c) = \frac{\alpha z + (1 - \alpha) A - \delta - \rho}{\sigma}$$

Finally,

$$g(h(k)) = \frac{h'(k)k}{h(k)} g(k) = \frac{\alpha z + (1 - \alpha)A}{z} g(k)$$

$$\begin{aligned}g(z) &= g(h(k)) - g(k) \\ &= \left[ \frac{\alpha z + (1 - \alpha)A}{z} - 1 \right] [z - \chi - n - \delta] \\ &= (1 - \alpha)(A/z - 1)[z - \chi - n - \delta]\end{aligned}$$

and

$$\begin{aligned}g(\chi) &= g(c) - g(k) \\ &= \frac{\alpha z + (1 - \alpha)A - \rho - \delta}{\sigma} - z + \chi + n + \delta \\ &= \varphi + \chi + z(\alpha/\sigma - 1)\end{aligned}$$

where  $\varphi = n + \delta + (1 - \alpha)A/\sigma - (\rho + \delta)/\sigma$ .

- In steady state,  $z_{ss} = A$  and  $\chi_{ss} = (1 - \alpha/\sigma)z_{ss} - \varphi$ .
- The  $\dot{\chi} = 0$  locus:

$$z = \frac{\varphi + \chi}{1 - \alpha/\sigma} \quad (55)$$

- A straight line with slope  $1 - \alpha/\sigma$ .
- For realistic parameter values,  $\alpha \simeq 0.3$  and  $\sigma \geq 1$  so that the slope is positive.
- $\dot{z} = 0$  has two solutions:  $z = A$  or  $\chi = z - n - \delta$ .
  - The second solution does not intersect the  $\dot{\chi} = 0$  line in the positive quadrant.
  - Note also that the production function restricts  $z \geq A$ .

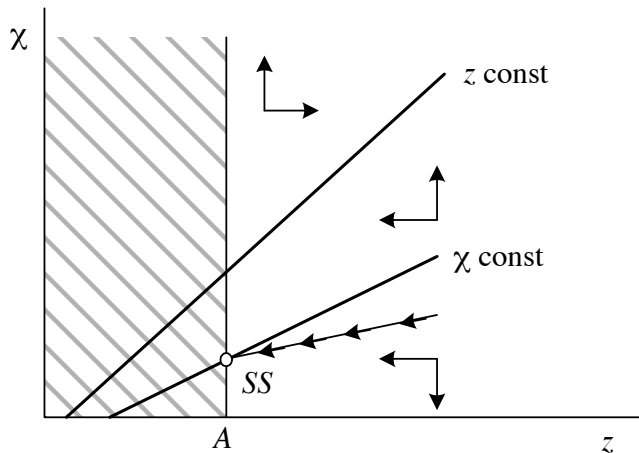
- Laws of motion:

$$\begin{aligned}\dot{z} &= (1 - \alpha)(A - z)(z - \chi - n - \delta) \\ g(\chi) &= \varphi + \chi + z(\alpha/\sigma - 1)\end{aligned}$$

- Steady state:

$$\begin{aligned}\dot{z} &= 0 : z = A \\ \dot{\chi} &= 0 : z = \frac{\varphi + \chi}{1 - \alpha/\sigma}\end{aligned}$$

# Phase Diagram with Endogenous Growth



# Phase Diagram with Endogenous Growth

- This system is **saddle-path stable**.
- If  $\chi_0$  is too small, then the trajectory crosses into the  $c < 0$  quadrant.
- If  $\chi_0$  is too large, then the trajectory takes off to the north-east.
  - This violates feasibility:  $\chi = c/k$  would grow without bounds.
- Both  $\chi$  and  $z$  converge monotonically to the steady state.
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The important point is the general approach for dealing with the dynamics of growing economies:

- 1 Write out the equilibrium conditions as usual.
- 2 Find conditions characterizing the balanced growth path.
- 3 Find ratios that are constant on the balanced growth path ( $\chi$  and  $z$ ).
- 4 Express the laws of motion of the economy in terms of these ratios.

An alternative approach is to transform the economy into stationary form before characterizing its equilibrium.

- Acemoglu, "Introduction to Modern Economic Growth," ch. 11.
- Krueger, "Macroeconomic Theory," ch. 9.
- Krusell, "Lecture notes for Macroeconomics I, 2004," ch. 8.
- Barro & Sala-i-Martin, ch. 1.3, 4.1, 4.2, 4.5.
- Jones, Larry E.; Rodolfo Manuelli (1990). "A convex model of equilibrium growth: Theory and policy implications." *Journal of Political Economy* 98(5): 1008-38.
- Lucas, Robert E. (1988). "On the Mechanics of Economic Development." *Journal of Monetary Economics* 22: 3-42.