Endogenous Growth: AK Model

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Endogenous Growth

Why do countries grow?

- A question with large welfare consequences.

We need models where growth is endogenous.

What drives growth in the data?

How could one answer this question empirically?

Outline

Necessary condition for sustained growth:

- MPK is bounded from below as $K \to \infty$
- It does not matter what K is.

The simplest endogenous growth model:

- a version of the standard growth model with **linear technology**
- $\dot{K} = sY - \delta K = sAK - \delta K$
- "AK model"

More interesting models have R&D driven growth

- new varieties
- quality improvements

All of these models have "AK" reduced forms.

Necessary Conditions for Sustained Growth

- How can growth be sustained without exogenous productivity growth?
- A necessary condition: **constant returns to the reproducible factors**.
  - The production functions for inputs that can be accumulated must be linear in those inputs.
  - Example: In the growth model, K would have to be produced with a technology that is linear in K
- This motivates a simple class of models in which
  1. only K can be produced and
  2. the production function is AK.
- This can be thought of as a reduced form for more complex models (we'll see examples).

2. Solow AK model

To see what is required for endogenous growth, consider the Solow model:

$$\dot{k} = sf(k) - (n+\delta)k \tag{1}$$

k could be anything that can be accumulated

- physical or human capital
- knowledge
- some combination of factors

The saving rate s can be endogenous

- its value does not matter (mostly)
- because s is bounded

Solow AK model

$$g(k) = sf(k)/k - (n+\delta) \tag{2}$$

Positive long-run growth requires:

- As $k \to \infty$ it is the case that

$$f(k)/k > n+\delta \tag{3}$$

- L'Hopital's rule implies (if f' has a limit):

$$\lim_{k\to\infty} f(k)/k = \lim_{k\to\infty} f'(k) \equiv A \tag{4}$$

Sustained growth therefore requires:

$$\lim_{k\to\infty} f'(k) \equiv A > n+\delta \tag{5}$$

Necessary Conditions for Sustained Growth

- This argument is more general than the Solow model.
  - It does not matter how s is determined.
- If $\lim_{k\to\infty} f'(k)$ exists, the production function has **asymptotic constant returns to scale**.

$$f(k) \to Ak + B \tag{6}$$

- It is fine to have diminishing returns for finite k.

Example

Diminishing returns with asymptotic AK:

$$f(k) = Ak + Bk^{\alpha} \tag{7}$$

- $0 < \alpha < 1$
- $f(k)/k \to A$ as $k \to \infty$

Example

CES production function with high elasticity of substitution:

$$F(K,L) = \left[\mu K^{\theta} + (1-\mu)L^{\theta}\right]^{1/\theta} \tag{8}$$

- $f(k) = \left[\mu k^{\theta} + 1 - \mu\right]^{1/\theta}$
- Elasticity of substitution: $\varepsilon = (1-\theta)^{-1}$.
- If $\theta > 0$ $[\varepsilon > 1]$,

$$f(k)/k \to \mu^{1/\theta} \tag{9}$$

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AK Solow Model

- In the Solow model, assume $f(k) = Ak$.
- Law of motion:

$$g(k) = sA - n - \delta \tag{10}$$

- Changes in parameters alter the growth rate of k.
- The model does not have any transitional dynamics: k always grows at rate $sA - n - \delta$.

AK Solow Model

It is not necessary to have constant returns in all sectors of the economy.

But the **reproducible factors** must be produced from reproducible factors with constant returns to scale.

Reproducible factors are inputs that are accumulated (here K).

Example

- c is produced from k with diminishing returns to scale:

$$c = [(1-s)Ak]^{\varphi} \tag{11}$$

  with $\varphi < 1$.
- The law of motion for k is unchanged (so is the balanced growth rate of k).
- This model still has a balanced growth path with a strictly positive growth rate
- but now c and k grow at different constant rates:

$$g(c) = \varphi\, g(k) \tag{12}$$

3. AK Growth Model

Setup

This model adds optimizing consumers to the Ak model.

Households maximize

$$\int_{t=0}^{\infty} e^{-(\rho-n)t} u(c_t)\, dt \tag{13}$$

subject to the flow budget constraint

$$\dot{k} = (r-n)k - c \tag{14}$$

There is no labor income because in the Ak world all income goes to capital.

AK model

For balanced growth we need

$$u(c) = c^{1-\sigma}/(1-\sigma) \tag{15}$$

The optimality conditions are the same as in the Cass-Koopmans model:

$$g(c) = (r-\rho)/\sigma$$

and the transversality condition

$$\lim_{t\to\infty} e^{-(\rho-n)t} u'(c_t) k_t = 0 \tag{16}$$

Firms

Firms maximize period profits.

The first-order condition is $r = A - \delta$.

There is no labor input and no labor income.

Equilibrium

An allocation: $c(t), k(t)$.

A price system: $r(t)$.

These satisfy:

1. Household: Euler, budget constraint (TVC).
2. Firm: 1 foc.
3. Market clearing:

$$\dot{k} = Ak - (n+\delta)k - c \tag{17}$$

Summary

Simplify into a pair of differential equations:

$$\dot{k} = (A-\delta-n)k-c \tag{18}$$
$$g(c) = (A-\delta-\rho)/\sigma \tag{19}$$

Boundary conditions: k_0 given and the TVC.

Of course, we could have simply taken the equilibrium of the standard growth model and replaced $f'(k) = A$ and $f(k) = Ak$.

Bounded utility

We need restrictions on the parameters that ensure bounded utility.

Lifetime utility is

$$\int_0^\infty e^{-(\rho-n)t}\,[c_0\,e^{g(c)t}]^{1-\sigma}\,dt/(1-\sigma) \tag{20}$$

Boundedness then requires that $n-\rho+(1-\sigma)g(c)<0$.

Instantaneous utility cannot grow faster than the discount factor $(\rho-n)$.

Transition dynamics

This model has no transitional dynamics.

Consumption growth is obviously constant over time.

To show that $g(k)$ is constant: we need to solve for $k(t)$ in closed form.

Details

Summary

The AK model has a very simple equilibrium.

1. The saving rate is constant.
2. All growth rates are constant.

This is very convenient, but also very limiting in many applications.

3.2. How to think about AK models?

In the data, there is at least one non-reproducible factor: labor.

Do models with constant returns to reproducible factors make sense?

The best way of thinking about AK models:

- a **reduced form** for a model with multiple factors
- there may be transition dynamics, but it does not matter if you are interested in long-run issues
- there may be fixed factors, but it does not matter if there are constant returns to reproducible factors.

Examples: AK as reduced form

1. Human capital: $F(K, hL)$ with K and h reproducible.
2. Externalities:
   2.1 Romer (1986). For the firm $F(k_i, l_i K) = K^{1-\alpha} k_i^{\alpha} l_i^{\theta}$
   2.2 Firms take K as given - diminishing returns to k_i.
   2.3 In equilibrium: $K = \sum k_i$ - constant returns to scale to K.
3. Increasing returns to scale at the firm level: $y = Ak^{\alpha} l^{1-\alpha}$
   3.1 A can be produced somehow - R&D.
   3.2 Need imperfect competition.

Summary

Sustained growth requires that inputs are produced with constant returns to reproducible inputs.

Then the model is (at least asymptotically) of the AK form: $\dot{K} = AK$.

The AK model is a reduced form of something more interesting.

Reading

- Acemoglu (2009), ch. 11.
- Krueger, "Macroeconomic Theory," ch. 9.
- Krusell (2014), ch. 8.
- Barro and Sala-i Martin (1995), ch. 1.3, 4.1, 4.2, 4.5.
- Jones and Manuelli (1990)

Digression: Solving for k(t) I

- Law of motion:

$$\dot{k}_t = (A - \delta - n)\, k_t - c_0 \ \exp\left(\frac{A - \delta - \rho}{\sigma} t\right) \tag{21}$$

- Solution to $\dot{x} = ax - b(t)$ is

$$x_t = x_0 e^{at} - e^{at} \int_0^t e^{-as} b(s)\, ds \tag{22}$$

- To verify:

$$\begin{aligned} \dot{x}_t &= ax_0 e^{at} - ae^{at} \int_0^t e^{-as} b(s)\, ds - e^{at} e^{-at} b(t) && (23) \\ &= ax_t - b(t) && (24) \end{aligned}$$

Digression: Solving for k(t) II

- Define

$$a = r - n = A - \delta - n > 0 \tag{25}$$

$$b = g_c = \frac{A - \delta - \rho}{\sigma} > 0 \tag{26}$$

- Then

$$k_t = k_0 \exp(at) - \exp(at) \int_0^t c_0 \exp([-a + b] s) \, ds \tag{27}$$

- Note:

$$\int_0^t e^{zs} ds = \frac{e^{zt} - 1}{z} \tag{28}$$

Digression: Solving for k(t) III

- Therefore:

$$k_t = k_0 e^{at} - \frac{c_0}{b-a} e^{at} \left[e^{(b-a)t} - 1 \right] \quad (29)$$

$$= \left[k_0 + \frac{c_0}{b-a} \right] e^{at} - \frac{c_0}{b-a} e^{bt} \quad (30)$$

- Now we show that $g(k)$ is constant: $k_t = k_0 e^{bt}$.
- Transversality:

$$\lim_{t - \infty} e^{(r-n)t} k_t = 0 \quad (31)$$

- Note that $a = r - n = A - \rho - n$.
- If $b > a$: $g(k) \to b > a$ and TVC is violated.
- So we need $b < a$.

Digression: Solving for k(t) IV

- With $b < a$ capital grows at rate a, unless the term in brackets is 0:

$$k_0 + \frac{c_0}{b-a} = 0 \tag{32}$$

- If $g(k) = a$, then $g\left(e^{-(r-n)t}k_t\right) = 0$ - because $a = r - n$.
  - That violates TVC.
- The only value of c_0 consistent with TVC is the one that sets the term in brackets to 0.
- It implies that k always grows at rate b.

Saving rate

- We can solve for c/k and the saving rate.

$$\begin{aligned} g(k) - g(c) &= [A - \delta - n - c/k] - (A - \delta - \rho)/\sigma = 0 \\ c/k &= A - \delta - n - (A - \delta - \rho)/\sigma \end{aligned}$$

- And the gross savings rate is

$$\begin{aligned} s &= (\dot{K} + \delta K)/AK \\ &= [g(K) + \delta]/A \\ &= [g(c) + n + \delta]/A \\ &= [(A - \delta - \rho)/\sigma + n + \delta]/A \end{aligned}$$

- The savings rate is high, if (σ, ρ or A) are low, or if n is high.

References I

Acemoglu, D. (2009): *Introduction to modern economic growth*, MIT Press.

Barro, R. and X. Sala-i Martin (1995): "Economic growth," *Boston, MA*.

Jones, L. E. and R. Manuelli (1990): "A Convex Model of Equilibrium Growth: Theory and Policy Implications," *Journal of Political Economy*, 1008–1038.

Krusell, P. (2014): "Real Macroeconomic Theory," Unpublished.

Romer, P. M. (1986): "Increasing returns and long-run growth," *The journal of political economy*, 1002–1037.