

# Review Questions: AK Models

Econ720. Fall 2009. Prof. Lutz Hendricks

## 1 Growth model with two capital goods

Consider the following endogenous growth model with two capital goods.

**Households:** There is a single, representative household who lives forever. Preferences over consumption streams are given by  $\sum_{t=0}^{\infty} \beta^t u(c_t)$ . Households own two capital goods,  $K_1$  and  $K_2$ . The income obtained from renting these capital goods to firms is their only source of income.

**Firms:** Production takes place in two sectors ( $i = 1, 2$ ). The resource constraints for sector 1 is

$$A_1 F_1(K_{11t}, K_{12t}) + (1 - \delta) K_{1t} = K_{1t+1} + c_t$$

where  $K_{ist}$  is the amount of capital of type  $s$  used in sector  $i$  and  $K_{st} = K_{1st} + K_{2st}$  is the total amount of capital good  $s$  used in both sectors. The resource constraint for sector 2 is similar, except that good 2 is not consumed:

$$A_2 F_2(K_{21t}, K_{22t}) + (1 - \delta) K_{2t} = K_{2t+1}$$

In each sector, a representative firm maximizes period profits. Assume that both production functions exhibit constant returns to scale.

(a) Define a solution to the firm's problem in each sector. Be careful to define the purchase and rental prices of the various goods consistently. Good 1 is the numeraire.

(b) State the household problem and define a solution.

(c) Define a competitive equilibrium. Make sure that the number of objects equals the number of equations.

(d) Consider the balanced growth path. Derive the balanced growth rates of  $c, k_s, r_s, p_s$  for  $s = 1, 2$ , where  $k_s = K_{s2}/K_{s1}$  is the input ratio in sector  $s$ ,  $r_s$  is the rental price of capital good  $s$ , and  $p_s$  is its purchase price. Assume log utility:  $u(c) = \ln(c)$ .

(e) Derive 7 equations that solve for 7 (constant) objects and thus define the balanced growth path.

(f) Using the 7 equations from (e), determine qualitatively how the balanced growth rate and prices change when  $A_1$  rises.

## 1.1 Answer: Growth model with two capital goods

To begin, we define prices.  $r_{st}$  is the rental price of capital good  $s$  in terms of good 1.

(a) The firm in sector  $i$  solves

$$\max A_i F_i(K_{i1t}, K_{i2t}) p_i - \sum_s r_{st} K_{ist}$$

The first-order conditions are  $r_s = A_i F_{is}(K_{i1}, K_{i2}) p_i$  for  $s = 1, 2$ . A solution is a pair  $(K_{i1t}, K_{i2t})$  which satisfies the 2 first order conditions.

(a) We anticipate that both capital goods must pay the same rate of return in equilibrium; call it  $R$ . Denote household wealth by  $a_t = p_{1t} K_{1t} + p_{2t} K_{2t}$ . Then the budget constraint is  $a_{t+1} = R_t a_t - c_t$ . The household problem is entirely standard with Euler equation  $u'(c_t) = \beta R_{t+1} u'(c_{t+1})$ . A solution is a sequence  $(c_t, a_t)$  which satisfies Euler equation and budget constraint (and a transversality condition).

(c) A competitive equilibrium is a set of sequences  $(c_t, a_t, K_{it}, K_{ist}, r_{it}, p_{it}, R_t)$  (13 objects) which satisfy:

- 2 household conditions (see b).
- 4 firm conditions (see a).
- Definition of the rate of return:  $R_{t+1} = [(1 - \delta) p_{st+1} + r_{st+1}] / p_{st}$ ;  $s = 1, 2$ . Giving up  $1/p_{st}$  units of good  $s$  today and investing the good as capital pays  $[(1 - \delta) p_{st+1} + r_{st+1}]$  units of the same good tomorrow. This rate of return must be the same for both goods (2 equations)
- Goods market clearing in both sectors (given in the question).
- Capital market clearing (also given):  $K_{st} = \sum_i K_{ist}$ .
- Definition of  $a_t$ .
- The normalization  $p_{1t} = 1$ .

There are  $2 + 4 + 2 + 2 + 2 + 1 = 14$  equations. One is redundant by Walras' law.

(d) We know that  $R$  must be constant, otherwise the consumption growth rate would not be. By the definition of  $R$ , this requires constant prices and rental prices. The quantities grow, all at rate  $\gamma$ .

(e) The Euler equation implies

$$1 + \gamma = \beta R.$$

The definition of  $R$  yields 2 additional equations

$$R = 1 - \delta + r_s/p_s.$$

The firms' first-order conditions are

$$r_s = A_i p_i F_{is}(K_{i1}, K_{i2})$$

Note that the marginal products (b/c of constant returns to scale) only depend on the inputs ratios  $k_i = K_{i2}/K_{i1}$ :

$$r_s = A_i p_i F_{is}(1, k_i)$$

(slightly abusing notation) (4 equations). With better notation: define  $f_i(k_i) = F_i(1, K_{i2}/K_{i1})$ . Then the firms' FOCs become

$$r_1 = A_1 p_1 [f_1(k_1) - f_1'(k_1) k_1] \tag{1}$$

$$r_2 = A_2 p_2 f_2'(k_2) \tag{2}$$

for  $i = 1, 2$ . This is entirely analogous to a model with capital and labor. Note that a higher  $k_i$  reduces  $f_i'(k_i)$  but increases  $f_i(k_i) - f_i'(k_i) k_i$ .

(f) Take the ratio of (2), (1) for both sectors and write this as  $r_1/r_2 = g_i(k_i)$ . Note that  $g_i'(k_i) > 0$ . From  $g_1(k_1) = g_2(k_2)$  it follows that  $k_1$  and  $k_2$  are positively related. Define this relationship as  $k_2 = h(k_1) = g_2^{-1}(g_1(k_1))$ . The positive relationship is not surprising. When  $r_1/r_2$  increases, firms in both sectors substitute towards the cheaper capital good. Now consider the condition

$$r_1 = A_1 [f_1(k_1) - f_1'(k_1) k_1] = A_2 f_2'(k_2) = r_2/p_2 \tag{3}$$

This can be written as

$$\frac{f_1(k_1) - f_1'(k_1) k_1}{f_2'(h(k_1))} = \frac{A_2}{A_1} \tag{4}$$

The LHS of (4) is increasing in  $k_1$ . It follows that a higher  $A_1$  reduces  $k_1$  and  $k_2$ . The intuition is that  $K_1$  is cheaper to produce and used relative more intensively. A lower  $k_2$  implies a higher  $r_1 = r_2/p_2$  by (3). Therefore  $R$  and the balanced growth rate  $\gamma$  must both increase.