Midterm Exam. Econ602. Spring 2008

Professor Lutz Hendricks

- Answer all questions.
- Write legibly! Write legibly! Write legibly!
- Write on only one side of each sheet.
- The total time is 1:45 hours.

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1 Education Costs

[40 points] Consider the following version of a standard growth model with human capital. The planner solves

$$\max \sum_{t=1}^{\infty} \beta^t u(c_t) \tag{1}$$

s.t.

$$k_{t+1} = (1-\delta) k_t + x_{kt} \tag{2}$$

$$h_{t+1} = (1-\delta) h_t + x_{ht} \tag{3}$$

$$c_t + x_{kt} + \eta x_{ht} = f(k_t, h_t) \tag{4}$$

with k_1 and h_1 given. Here c is consumption, k is physical capital, h is human capital, and η is a constant representing education costs. Assume that the production function is Cobb-Douglas:

$$f(k,h) = z k^{\alpha} h^{\varepsilon} \tag{5}$$

where z is a constant technology parameter and $\alpha + \varepsilon < 1$.

(a) Derive the first-order condition for the planner's problem using Dynamic Programming. Define a solution in sequence language and in functional language.

(b) Solve for the steady state levels of k/h and k.

(c) Characterize the impact of cross-country differences in education costs (η) on output per worker in steady state. In particular, calculate the ratio of outputs per worker for two countries that only differ in their η's.

2 Relative Wealth Preferences

[60 points] Consider the following version of the growth model in continuous time.

Demographics: There is one representative household.

Preferences:

$$\int_0^{\infty} e^{-\rho t} \left[U(c_t) + V\left(k_t/\bar{k}_t\right)\right] dt \tag{6}$$

Endowments: The household starts with k_0.

Technology:

$$\dot{k}_t = f(k_t) - c_t \tag{7}$$

Government budget constraint: The government taxes consumption at rate τ_c and lump-sum rebates the revenues R_t to the household.

$$R_t = \tau_c c_t \tag{8}$$

Household budget constraint:

$$\dot{k}_t = f(k_t) - (1+\tau_c) c_t + R_t \tag{9}$$

Notation: c: consumption, k: capital, $\bar{k}$: average capital in the economy.
Assumptions: U, V, f are strictly increasing and strictly concave. $f'(0) = \infty$. $f'(\infty) = 0$.

1. State the household's current value Hamiltonian and derive the first-order conditions. Do not yet substitute out the co-state. Define a solution to the household problem.

2. Define a competitive equilibrium.

3. Derive an equation that implicitly solves for the steady state capital stock.

4. Draw the phase diagram. Start with $\dot{k} = 0$ and discuss its shape.

5. Derive $\dot{c} = 0$ and discuss its slope / intercept. For which values of k does $\dot{c} = 0$ have a solution? Hint: It is easier to write down $\dot{\lambda} = 0$, where λ is the co-state. Then use the fact that $\dot{\lambda} > 0$ implies $\dot{c} < 0$.

6. Assume that $\dot{c} = 0$ is concave,

   $$\partial^2 c / \partial k^2 |_{\dot{c}=0} < 0 \tag{10}$$

   and that it intersects $\dot{k} = 0$ twice. Discuss the stability properties of the two steady states.

3 Answers

3.1 Answer: Education Costs

(a) The planner's Bellman equation is

$$\begin{aligned} V(k,h) &= \max u(c) + \beta V\left((1-\delta)k + x_k, (1-\delta)h + x_h\right) \\ &\quad + \lambda\left[f(k,h) - c - x_k - \eta x_h - g\right] \end{aligned}$$

First-order conditions:

$$\begin{aligned} u'(c) &= \lambda \\ \beta V_k(.') &= \lambda \\ \beta V_h(.') &= \eta\lambda \end{aligned}$$

Envelope conditions:

$$\begin{aligned} V_k(k,h) &= \beta V_k(k',h')(1-\delta) + \lambda f_k(k,h) \\ V_h(k,h) &= \beta V_h(k',h')(1-\delta) + \lambda f_h(k,h) \end{aligned}$$

Simplify to obtain an Euler equation, which is perfectly standard:

$$u'(c) = \beta u'(c')\left[1-\delta + f_k(k',h')\right]$$

In addition, there is a second Euler equation

$$u'(c) = \beta u'(c')\left[1-\delta + f_h(k',h')/\eta\right]$$

which can be made into a static condition

$$1-\delta + f_k(k',h') = 1-\delta + f_h(k',h')/\eta$$

A solution consists of sequences c, k, h, x_k, x_h that solve 2 laws of motion, 1 feasibility condition, 2 first-order conditions.

(b) Imposing functional forms: $k/h = \eta\alpha/\varepsilon$. The steady state capital stock is determined by

$$1/\beta = z\alpha k^{\alpha-1+\varepsilon}\left[\varepsilon/(\alpha\eta)\right]^{\varepsilon} + 1 - \delta$$

Steady state output is

$$f(k_{ss},h_{ss}) = z k_{ss}^{\alpha+\varepsilon}\left[\varepsilon/(\alpha\eta)\right]^{\varepsilon}$$

(c) An increase in η reduces both k and h in steady state. How much do education costs affect output per worker? The output ratio of two countries is

$$\frac{f^A}{f^B} = \left(\frac{k_{ss}^A}{k_{ss}^B}\right)^{\alpha+\varepsilon}\left(\frac{\eta_B}{\eta_A}\right)^{\varepsilon}$$

The ratio of capital stocks can be derived from the steady state k equation:

$$k_{ss}^A/k_{ss}^B = \left(\eta_A/\eta_B\right)^{\varepsilon/(\alpha+\varepsilon-1)}$$

Finally,

$$f^A/f^B = \left(\eta_A/\eta_B\right)^{\varepsilon/(1-\alpha-\varepsilon)}$$

3.2 Answer: Relative wealth preferences

1. This question is based on Wen-ya Chang, "Relative wealth, consumption taxation, and economic growth," *Journal of Economics* 2006, 88(2): 103-29.

$$H = U(c) + V\left(k/\bar{k}\right) + \lambda\left[f(k) - R - (1+\tau_c)c\right] \tag{11}$$

FOC:

$$U'(c) = \lambda(1+\tau_c) \tag{12}$$
$$V'\left(k/\bar{k}\right)/\bar{k} + \lambda f'(k) = \rho\lambda - \dot{\lambda} \tag{13}$$

Solution: c, k, λ that solve 2 FOCs, budget constraint, TVC: $\lim e^{-\rho t}\lambda_t k_t = 0$.

2. CE: $c, k, \bar{k}, \lambda, R$ that solve: household (3), government budget constraint, goods market clearing

$$f(k) = c + \dot{k} \tag{14}$$

and the identity $k = \bar{k}$.

3. Steady state: Set $\dot{\lambda} = 0$ and $f(k) = c$:

$$V'(1)/k = \frac{U'(f(k))}{1+\tau_c}\left[\rho - f'(k)\right] \tag{15}$$

Relative to the case of $V = 0$, steady state capital is higher.

4. Phase diagram. $\dot{k} = 0$ requires $c = f(k)$. Simply plot the production function. Higher c reduces $\dot{k}$.

5. $\dot{\lambda} = 0$ requires

$$\lambda = \frac{U'(c)}{1+\tau_c} = \frac{V'(1)}{k\left[\rho - f'(k)\right]} \tag{16}$$

Since $\lambda > 0$, this is only defined for sufficiently high k such that $\rho - f'(k) > 0$. The slope of $k\left[\rho - f'(k)\right]$ is $\left[\rho - f'(k)\right] - kf''(k) > 0$. Therefore, $\partial\lambda/\partial k|_{\dot{c}=0} < 0$ and $\partial c/\partial k|_{\dot{c}=0} > 0$. From

$$\dot{\lambda} = \lambda\left[\rho - f'(k)\right] - V'(1)/k \tag{17}$$

it follows that higher λ raises $\dot{\lambda}$. Thus, higher c raises $\dot{c}$.

6. Draw the phase diagram. It should look like figure 1. The lower steady state is saddle path stable. To show this, argue that any path not leading to the steady state would violate a boundary condition. The tricky part here: showing that the area where $\dot{c} < 0$ and $\dot{k} > 0$ cannot be reached. It would violate TVC. To see this write

$$\begin{aligned} g(\lambda) + g(k) &= \rho - \frac{V'(1)}{\lambda k} - f'(k) + \frac{f(k) - c}{k} \\ &= \rho - \frac{V'(1)/\lambda + c}{k} + \frac{f(k) - f'(k)k}{k} \end{aligned}$$

Note that for large enough k, $g(\lambda) + g(k) > \rho$. This happens because $V'(1) + c$ are falling and $f(k) - f'(k)k$ [which equals labor income] is rising over time.

The upper steady state is unstable. It has ever rising c and k. Can this exist? A path close to $\dot{\lambda} = 0$ satisfies the TVC, which implies

$$V'(1) = \lambda k \left[\rho - f'(k)\right]$$

or asymptotically $\lambda k \to V'(1)/\rho$. If parameters are such that $c < f(k)$ and $c > u'^{-1}\left(\frac{V'(1)(1+\tau_c)}{\rho k}\right)$ can be simultaneously satisfied as $k \to \infty$, then such an equilibrium exists. If not, then the phase diagram does not look like figure 1.

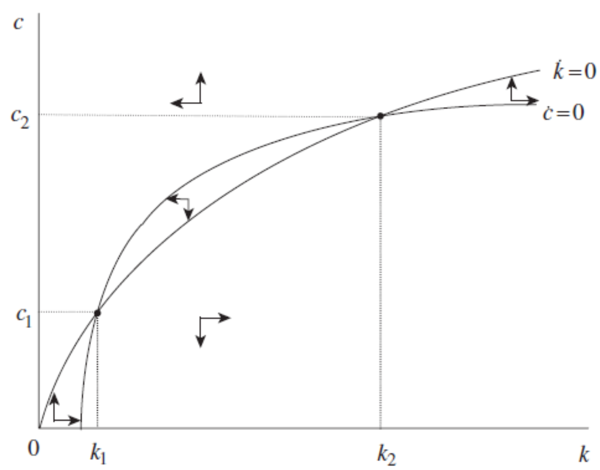

Figure 1: Phase diagram