The Solow Model

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The Solow Model

- In the production model capital is exogenous.
- We learn how much capital matters, but we cannot learn why some countries lack capital.
- We need a model with capital accumulation (investment, saving).
- That also answers the question: Does capital drive long-run growth?
- We add just one piece to the production model:
  - an equation that describes how capital is accumulated over time through saving.

Model Elements

- The world goes on forever.
  - Time is indexed by the continuous variable t.
- There is **one good** which is produced from capital K and labor L.
- The **aggregate production function** is

$$
\begin{aligned}
Y(t) &= F[K(t), L(t), A(t)] \qquad (1) \\
&= K(t)^{\alpha} \; [A(t) \; L(t)]^{1-\alpha}
\end{aligned}
$$

  - A is an index of the state of "technology" (anything that makes people more productive over time).
  - A grows over time for reasons that are not modeled (a major shortcoming of the model).

Model Elements

- L grows over time at rate n:

$$L(t) = L(0)\, e^{nt}$$

- **Normalize** $L(0) = 1$
  - Why can I do that?
- Output is divided between consumption and gross investment:

$$Y(t) = C(t) + I(t) \tag{2}$$

Capital Accumulation

- Investment contributes to the capital stock:

$$\dot{K}(t) = I(t) - \delta K(t) \tag{3}$$

- $\dot{K}(t) = dK(t)/dt$ is the time derivative of $K(t)$.
  - the change in K per "period".
- δ is the rate of depreciation.

Capital Accumulation: Discrete Time

- To better understand the law of motion for K, we look at a discrete time version.
- Enter the period with capital stock $K(t)$.
- Lose $\delta K(t)$ to depreciation.
- Produce $I(t)$ new machines.
- Change in the capital stock: $K(t+1) - K(t) = I(t) - \delta K(t)$.

Capital Accumulation: Discrete Time

- Now we look at shorter time periods of length Δt.

$$K(t+\Delta t) - K(t) = [I(t) - \delta K(t)] \times \Delta t \tag{4}$$

or

$$\frac{K(t+\Delta t) - K(t)}{\Delta t} = I(t) - \delta K(t) \tag{5}$$

- The change in capital per unit of time is given by investment minus depreciation.
- Let $\Delta t \to 0$ then $\frac{K(t+\Delta t)-K(t)}{\Delta t} \to$

Choices

- This is a closed economy. Saving equals investment: $S(t) = I(t)$.
- Note: All of the above is simply a description of the production technology.
  Nothing has been said about how people behave.
- People make two fundamental choices (in macro!):

1. How much to save / consume.
2. How much to work.

Choices

- **Work**: we assume L is fixed.
- **Consumption** / saving: We assume that people save a fixed fraction of income:

$$C_t = (1 - s)\, Y_t \tag{6}$$

- Equivalently:

$$I_t = sY_t \tag{7}$$

Model Summary

1. Cobb-Douglas production function

$$Y(t) = K(t)^{\alpha} \ [A(t) \ L(t)]^{1-\alpha} \tag{8}$$

2. Law of motion for capital:

$$\dot{K}(t) = I(t) - \delta K(t) \tag{9}$$

3. Constant population growth: $L(t) = L(0) \ e^{nt}$.
4. Constant productivity growth: $A(t) = A(0) \ e^{\gamma t}$.
   For now: $\gamma = 0$.
5. Constant saving rate: $I(t) = s \ Y(t)$.

The Law of Motion for Capital

Solving the Model

- Even this simple model cannot be "solved" algebraically.
- That is, we cannot write the endogenous variables as functions of the parameters.
- This is almost never possible in **dynamic** models.
  - Dynamic means: there are many time periods.
  - All interesting macro models are dynamic.
- What we can do is
  1. graph the model and trace out qualitatively what happens over time.
  2. solve the model for the long-run values of the endogenous variables (e.g. K_t as $t \to \infty$).

The Solow Diagram

- We condense the model into a single equation in K.
- It will be a dynamic equation that tells us how K changes over time as a function of K_t.
- Then we graph the equation.

The Solow Equation

- Start from the law of motion:

$$\dot{K}(t) = I(t) - \delta \ K(t) \tag{10}$$

- Impose constant saving:

$$\dot{K}(t) = s \ Y(t) - \delta \ K(t) \tag{11}$$

- Impose the production function:

$$\dot{K} = sK^{\alpha}[AL]^{1-\alpha} - \delta K \tag{12}$$

Per capita growth

- We express everything in per capita terms. E.g., $y = Y/L$, etc.
- Output per capita is derived from $Y = K^{\alpha}\ [AL]^{1-\alpha}$:

$$y = (K/L)^{\alpha}\ A^{1-\alpha} \tag{13}$$

- Let's ignore technical change for now and set A constant.
- Now we have

$$\dot{K}/L = s\underbrace{A^{1-\alpha}(K/L)^{\alpha}}_{Y/L} - \delta K/L \tag{14}$$

or

$$\dot{K}/L = sA^{1-\alpha}k^{\alpha} - \delta k \tag{15}$$

The law of motion for capital

- Claim: $\dot{k} = \dot{K}/L - nk$.
- The law of motion can then be written as

$$\dot{k} = sA^{1-\alpha}k^{\alpha} - (n+\delta)k \tag{16}$$

- Intuition:
    - Suppose you invest nothing $(s = 0)$. Then K drops by δ each period due to depreciation.
    - K/L declines even more because the number of people increases by n each period.

Proof of the law of motion

$$\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta \tag{17}$$

$$k = K/L \tag{18}$$

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \tag{19}$$

$$= s\frac{y}{k} - \delta - n \tag{20}$$

Digression: What modern macro would do

- Modern macro would replace the constant saving rate with an **optimizing household**.
- Households **maximize utility** of consumption, summed over all dates.
- They choose time paths of $c(t)$ and $K(t)$.
- The **saving rate** would be endogenous and depend on
  - the interest rate (marginal product of K)
  - productivity
  - population growth
  - **expectations** of all future variables.
- What do we gain from this complication?

Factor prices

- Assume that (K, L) are paid their marginal products:

$$q = \partial F / \partial K \tag{21}$$
$$w = \partial F / \partial L \tag{22}$$

- q is the rental price of K, it is **not the interest rate**.
- What is an interest rate?

The Interest Rate

Definition

The interest rate answers the question:

:

The Interest Rate

What is the interest rate in the Solow model?

- Rent 1 unit of c to the firm at t.
- At $t+1$ receive:
  1. q_{t+1} in rental income;
  2. $1-\delta$ units of undepreciated capital.
- The interest rate is therefore: $1+r_{t+1} = q_{t+1} + 1 - \delta$.

The Solow Diagram

Analyzing the Solow Model

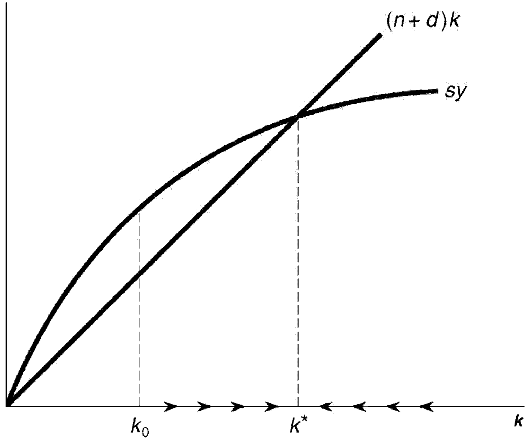
What are the properties of the Solow model?

- Why do economies grow over time?
- Does the economy settle down in the long-run?
- What are the long-run and short-run effects of changes in behavior?

To answer that: Plot the law of motion for k:

$$\dot{k}(t) = s \underbrace{k(t)^{\alpha} A^{1-\alpha}}_{y(t)} - (n+\delta) k(t)$$

The Solow diagram



The capital stock converges over time to the **steady state**, k^*.

The steady state

Definition

A **steady state** is a situation where all variables are constant over time (in per capita terms).

- In the Solow model:
  - Capital per worker is constant: $\dot{k} = 0$.
  - The steady state capital stock solves:

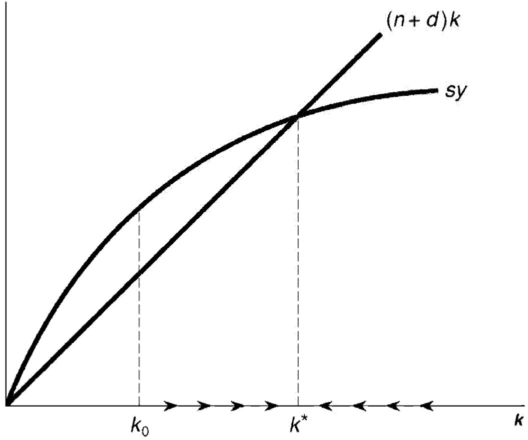
$$sf(k^*) = (n + \delta)\, k^* \tag{23}$$

    (Intuition?)

- The graph shows that there is a **unique** steady state (because we assumed that f has nice properties).
- Parameters that determine the steady state: n, δ, s and parameters of f.

Comparative statics (or dynamics)

What happens if households save more?

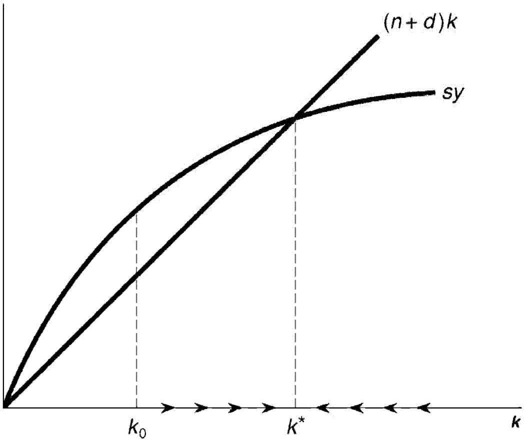


Plot the time paths of output and interest rates.

Reality Check

- The model says: more investment (or **lower consumption**) generates a period of **faster** growth.
- Isn't everybody saying: the U.S. is in a recession (slow growth) because consumption is too low?
- How does the contradiction get resolved?
- Where is the effect of lower consumption demand in the Solow model?
- Where is the demand side anyway?

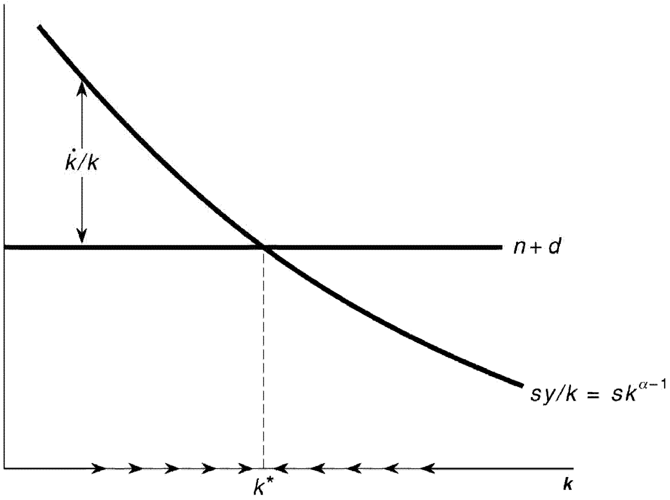
What happens if there is a baby boom?



Economic Growth

- Why do countries grow?
- In the Solow model:
  - Growth can only occur along a **transition path**.
  - There is **no long-run** growth in GDP per worker $(y = Y/L)$.
- But growth slows as the economy approaches the steady state.
- To see this, write the law of motion for k as

$$\dot{k}/k = g(k) = sy/k - (n + \delta)$$


$\dot{k}/k$
$n + d$
$sy/k = sk^{\alpha-1}$
k^*
k

Why does investment not sustain growth?

- The problem is the diminishing MP_K.
- Giving up one unit of C today yields $MP_{K'} - \delta$ in additional output tomorrow.
- As k grows, MP_K eventually falls below δ:
  - Additional investment no longer even pays for its own depreciation.
- Sustained growth through capital accumulation requires that MP_K stays above δ, even as k grows without bounds.

Technical Change

Technical change

- To sustain long-run growth of y the Solow model requires **technical change**.
  - Technical change is modeled as shifting the production function up.
  - Productivity grows: $g(A) > 0$.
- Modern models treat A as the product of innovation.
- Here: A is exogenous.
- Assume that technical change is **labor augmenting**: $Y = F[K, AL]$.
  - Otherwise, the model is not consistent with the Kaldor facts (not obvious, but true).

Law of motion with technical change

- Production function $Y = K^{\alpha} \ (AL)^{1-\alpha}$
- Law of motion (unchanged): $\dot{K} = sY - \delta K$
- or

$$g(K) = s \ Y/K - \delta$$
$$= s \ y/k - \delta$$

- Consider variables divided by AL: $\bar{y} = Y \,/\, AL, \bar{k} = K \,/\, AL,$ etc.
  - These will be constant in "steady state".
- By the growth rate rule:

$$g\left(\bar{k}\right) = g(K) - g(A) - n$$
$$= s \ \bar{y}/\bar{k} - g(A) - n - \delta$$

- This is the key equation for the Solow model with technical change:

$$d\bar{k}/dt = s\,\bar{y} - (n + \delta + g(A))\,\bar{k} \qquad (24)$$

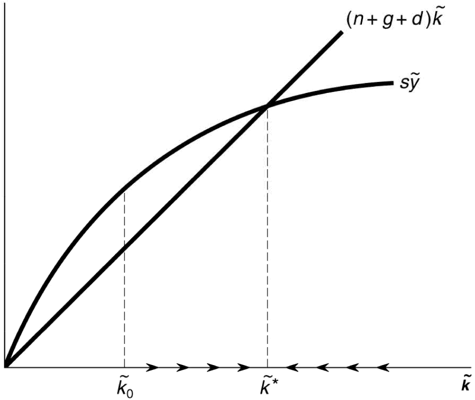
What has changed?

The model with technical change looks exactly like the previous model, except:

1. All variables are "detrended" (divided) by AL.
2. The steady state has per capita variables growing at rate $g(A)$.
3. The law of motion contains an additional $g(A)$ term.

The model has a steady state in the "detrended" variables $(\bar{k}, \bar{y})$.
It has a balanced growth path in per capita variables (k, y).
Definition: A **balanced growth path** is a situation in which all variables grow at constant rates.

The Solow diagram

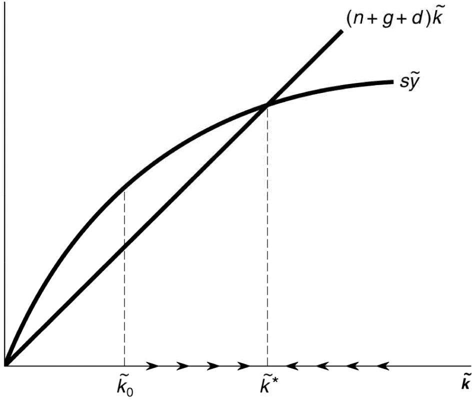


This is essentially the same diagram as without technical change, except:

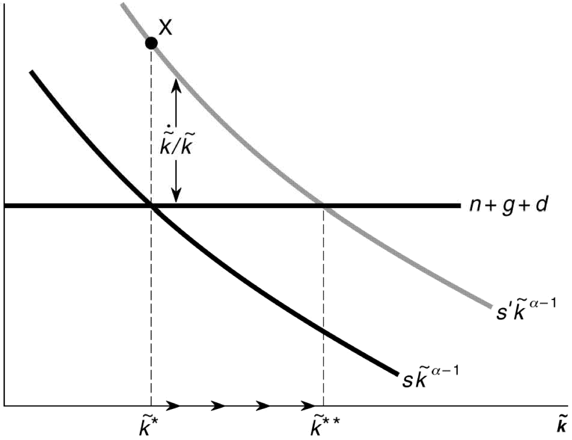
- variables are detrended.
- an additional g term appears in the straight line.

Comparative statics: higher saving rate

The Solow diagram is familiar:



Transitional dynamics



The Principle of Transition Dynamics

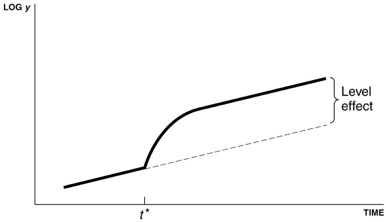
Fact

In the Solow model, the farther away the economy is from its steady state, the faster it grows (or shrinks)

What is the intuition?

Policies have level effects

A key implication of the Solow model: Policies, such as taxes, do not affect the long-run growth rate.
The growth rate rises on the transition to the new steady state, then levels off to $g(A)$.



Important Points

- The Solow model reveals how choices (saving, fertility) affect capital and output (levels and growth).
- Capital cannot sustain long-run growth.
  - the reason: diminishing returns
- Therefore policies have level effects.
- In the short run: countries grow fast when they are far below their steady states.
- In the long run: growth is determined by productivity improvements.

Final Example

Modify the Solow model by assuming that the production function is given by

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha} - X$$

1. How does the Solow diagram change?
2. How many steady states are there?
3. Which ones are stable?

Reading

- Jones, Introduction to Economic Growth, ch. 2
- Romer, D. (2012). Advanced macroeconomics (4 ed.). McGraw-Hill/Irwin, ch. 1
- Barro, R. J., & Sala-i-Martin, X. (2003). Economic growth (2nd ed.), ch. 1.2