# Static Optimization

**Econ602. Spring 2007. Lutz Hendricks**

## 1. Unconstrained Optimization

We are given a function $f(x)$. We want to find the $x^*$ that maximizes $f$.

Assumption 1: $f$ is twice differentiable.

### 1.1 Case 1: $x$ is a scalar

Recipe:

- A *necessary* condition for $x^*$ to be a maximum is $f'(x^*) = 0$. That is, *if* $x^*$ is a max, *then* the derivative at $x^*$ must be 0. We call such an $x^*$ a *critical point*. The condition is also called a *first-order* condition.
- A *sufficient condition* for $x^*$ to be a maximum is that it is a critical point *and* $f''(x^*) < 0$. That is, *if* $x^*$ satisfies both equations, *then* it must be a maximum. The condition is also called a *second-order* condition.

Comments:

- $f$ may have many critical points. The sufficient conditions helps us pick the maximum out of the set of all critical points.
- Both conditions only establish that $x^*$ is a *local* max. That is, there is no point "close to" $x^*$ that has a higher $f$-value. But there may be other points not "close to" $x^*$ that do. In that case there must be multiple points fulfilling both conditions and we simply calculate which of these has the highest $f$-value.
- If $f$ is strictly concave, there is only one critical point and this must be the global max. The reason is that $f''(x) < 0$ everywhere (that is what it means to be strictly concave). Thus, there can only be one point where $f'(x) = 0$ and that satisfies the sufficient condition automatically.

Figure 1 illustrates a case with 3 critical points, two local maxima and (of course) one global maximum.

### 1.2 Case 2: x is a vector

The fact that $x$ is a vector does not change the necessary condition at all. We can vary the components of $x$ *independently* and the necessary condition simply says it must be impossible to increase the objective function $f$ by varying the components one-by-one. Thus:

*If* $x^*$ is a maximum, *then* $\partial f / \partial x_i (x^*) = 0$ for each $i$.

Sufficient conditions are more complicated than in the scalar case. But it remains true that a globally strictly concave function has a unique global maximum at the unique critical point.

# 2. Optimization with Equality Constraints

We seek to solve $\max f(x)$ subject to $g_i(x) = 0$.

Recipe:

- Write the Lagrangean: $L(x,\lambda) = f(x) - \sum_i \lambda_i g_i(x)$, where the $\lambda_i \geq 0$ are unknown constants, called the "Lagrange-multipliers". This essentially transforms the constrained optimization problem into an unconstrained one: Choose $x$ and the $\lambda_i$ so as to maximize $L$.
- The necessary conditions for a maximum are: $\partial L / \partial x_i (x^*) = 0$ for all $i$. In addition, $x^*$ must of course satisfy the constraints $g_i(x^*) = 0$.
- Ignore sufficient conditions for now.

Comments:

- Logically, the necessary conditions say: "If you know the maximum, then it must satisfy the necessary conditions." But the value of this is to find candidates for $x^*$ by solving the system of necessary conditions together with the constraints for $(x, \lambda)$ pairs. If there is a maximum, *one* of the solution pairs must be it.
- Sufficient conditions (SOC's) are needed to pick the maximum out of possibly many critical points. An important special case is *concave programming*: If the objective function is strictly concave and all constraints are strictly convex, then there is a unique critical point which is a global maximum (Dixit 1990, ch. 7). More general SOC's are ugly.
- Lagrange multipliers have an important interpretation. Relaxing constraint $i$ by a small amount $\varepsilon$ can be shown to increase the maximum value $f(x^*)$ by $\lambda_i \varepsilon$ (Dixit 1990, ch. 4).

Examples:

- Dixit (1990), examples and exercises in chapter 2

## 2.1 Reading

Dixit (1990), ch. 2. Kreps appendix.

Barro and Sala-i-Martin (1995), appendix 1.3. A heuristic introduction. Part 1.3.10 provides a useful recipe-style summary.

# 3. Static Optimization with Inequality Constraints

The problem is $\max\ f(x_1,...,x_n)$ subject to

$$g_i(x_1,...,x_n) \le a_i\ ;\ \ i = 1, \ldots, m$$

To simplify notation, write $x$ for the vector $(x_1, \ldots, x_n)$. Assume that the constraints $g_i(.)$ are twice continuously differentiable and $f(.)$ is strictly concave and twice continuously differentiable (probably more than we need).

### 3.1.1 Kuhn-Tucker theorem

If $x^*$ is a solution to this problem, then there exist $m$ Lagrange multipliers $\mu_i$ such that:

(a) $\partial\Gamma / \partial x_i = 0$

(b) $g_i(x^*) \le a_i$, $\mu_i \ge 0$

(c) $\mu_i [a_i - g_i(x^*)] = 0$

Here, $\Gamma$ is the same Lagrangean as used for equality constrained problems, except that it now matters whether the constraints are added or subtracted:

$$\Gamma = f(x) + \sum_{i=1}^{m} \mu_i [a_i - g_i(x)] .$$

A trick to remember whether to add or subtract the constraints: Think of $\Gamma$ as an indirect utility function. Adding a bit to $a_i$ relaxes a constraint and must increase utility. Note that the conditions are very similar to those for equality constrained problems.

Condition (*c*) is called the *complementary slackness* condition. It states that the multiplier $\mu_i$ must be zero, if constraint $i$ is slack (holds with strict inequality). However, if the multiplier is strictly positive, then constraint $i$ must bind (hold with equality).

### 3.1.2 Reading

Kreps appendix. Dixit (1990), ch. 3. BS appendix. Sundaram (1996).

# 4. Reading:

Sundaram (1996).

Kreps, David. *A Course in Microeconomic Theory*. The appendix contains an excellent, very readable introduction, especially for Kuhn-Tucker problems. Recipe style. Incidentally, the entire book is excellent.

Dixit, Avinash K. *Optimization in Economic Theory*. Oxford University Press. 1990. A quite good introduction to static optimization with a lot of intuition and examples. Starts out with the simplest cases, but covers many important extensions.

## 5. Figures

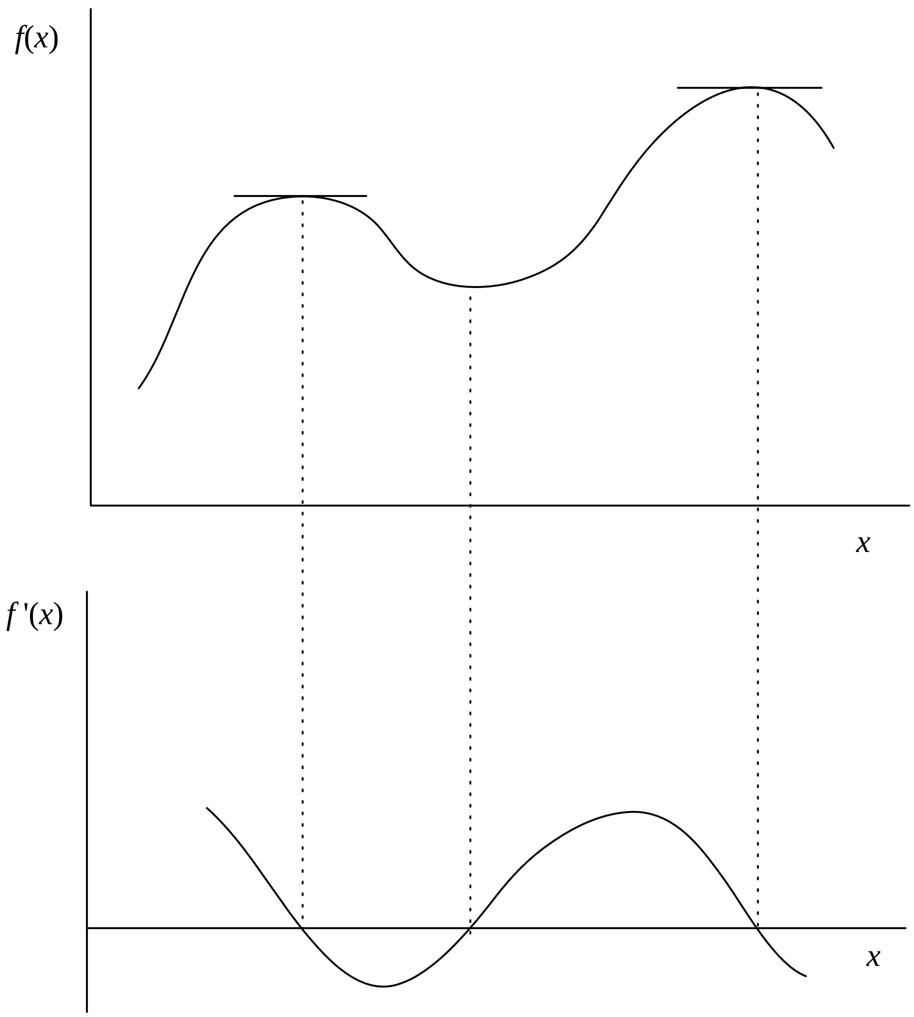


Figure 1. Unconstrained Optimum